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ON THE SPECIFIC DYNAMIC ACTION OF PROTEIN

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In a previous communication[1] evidence was presented that the normal excretion of nitrogen by the mammalian kidney is attended by a relatively great expenditure of energy, which in man amounts to 6–11 kg. calories per gram of nitrogen. The work of renal excretion is, therefore, an important element in the specific dynamic action of protein.

Current views on the specific dynamic action of protein, whether derived from Rubner or Voit, look to one process only as responsible for the increased metabolism following the ingestion of protein. In this Zuntz,[2] who first ascribed the specific dynamic action of protein to the work of digestion and absorption, and later to the work of excretion of nitrogen, held in this respect essentially a similar position.

The purpose of the present communication is to show that the course of the specific dynamic action of protein parallels the course of nitrogen excretion; and in the well-nourished animal is the result of at least two processes, of which one is the work imposed upon the kidney, and the other is what may be called the "specific dynamic action proper," due to the metabolism other than excretion of the nitrogen, and of the carbon.

In the majority of studies of the specific dynamic action of protein, the recorded observations, upon which the stated conclusions are based, cover a period in which a fraction only of the ingested nitrogen was accounted for, and the magnitudes of the increases in metabolism obtained have been referred to the total weight, or amount of nitrogen ingested. If the specific dynamic effects of two substances, glycine and glutamic acid for example, are compared, one of which is absorbed and metabolized quickly, the other very slowly, and the data upon which the comparison is based are derived from observations during the first few hours, only, and if the comparison is referred also to the amount ingested, not metabolized, it is obvious that the substance metabolized more quickly, all other factors being the same, will appear to exert the larger specific dynamic effect. Nevertheless, differences in rate of metabolism often have

not been taken into account. When this factor is not omitted from consideration, the explanation of some hitherto anomalous phenomena becomes evident, viz.: the neutralization of the specific dynamic effect of glycine or alanine when administered with protein; the apparently low specific dynamic effect of glutamic acid; and the reduction of the specific dynamic action of protein in cases of endocrine disturbance.

A correlation between the amount of additional urea and ammonia excreted and the extent of the specific dynamic action has been controverted by Lusk and his pupils. Accordingly we have set out in figures 1, 2 and 3, and tables 1, 2 and 3, the data obtained on the dog, by Rapport,[3,8] Weiss and Rapport,[4] and Rapport and Beard[5] which, we believe,

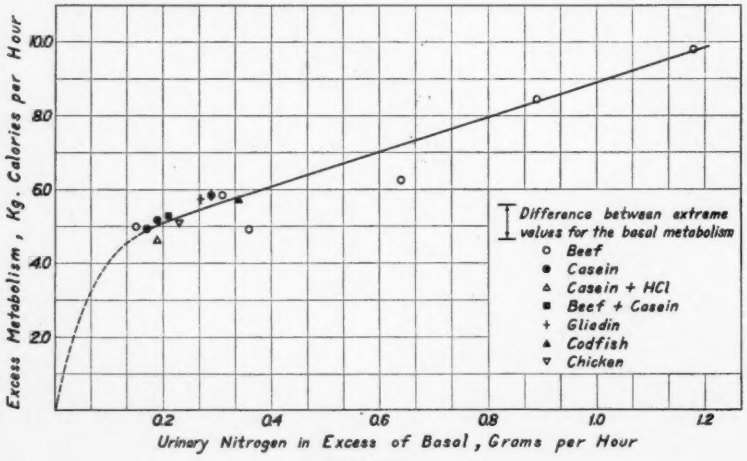

FIGURE 1

clearly demonstrate this correlation, although Rapport, and Weiss and Rapport stated that there is no clearly demonstrable relation between the nitrogen eliminated and the increase in metabolism.

The data have been arranged in three groups: one containing the results obtained with proteins only; another, the data on glycine, alanine and gelatin; and one containing the results with glycine, gelatin, and total and fractionated hydrolysates of gelatin. The later observations of Rapport and Beard[5] show the necessity for not including gelatin with the other proteins if just comparisons are to be made. Since a number of amino acids exert a greater specific dynamic effect, per gram of nitrogen excreted, than glycine or alanine, it is to be expected that the effect of most proteins in general will be greater than that of these two simple amino acids.

Gelatin, on the other hand, contains 26 per cent glycine, 9 per cent

alanine, and 25 per cent of the slowly absorbed dicarboxylic amino acids and leucine. It is low in phenylalanine and is lacking in tyrosine; both of these amino acids have higher specific dynamic effects per mol metabolized than glycine or alanine. It belongs, therefore, in respect of its specific dynamic action in the first few hours, more properly with glycine and alanine than with the proteins. The results given in tables 1 and 2 and figures 1 and 2 were obtained on the same dog; those in table 3, and figure 3, were obtained some time later on a different dog.

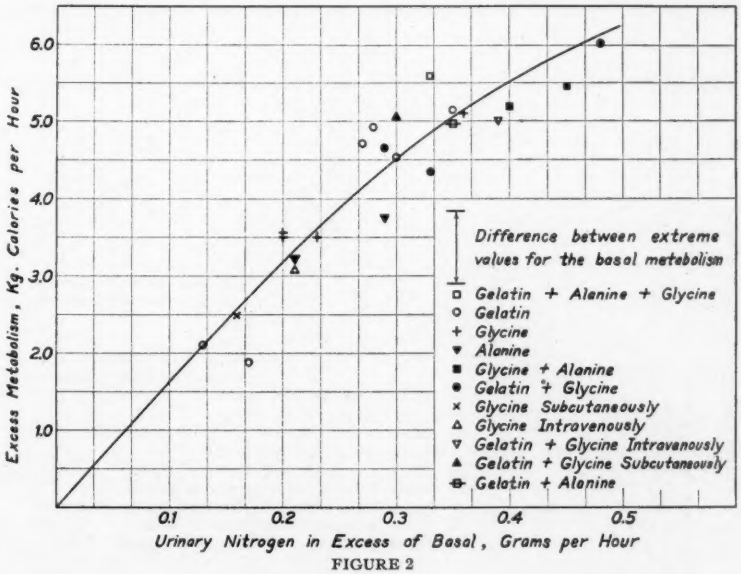

FIGURE 2

Figures 1, 2 and 3, and tables 1, 2 and 3 show the correlation between nitrogen excretion and excess metabolism. In view of the difficulty of the experimental technique, and of the variability of animals, the concordances of the results obtained by Rapport, Weiss and Rapport, and Rapport and Beard, is remarkably good. Figure 1, which is the most extended of the three curves, shows that the relationship between excess metabolism and excess nitrogen excreted is not defined by a straight line passing through the origin. The curve seems, rather, to begin from the origin as a steep curve, and to pass later into a straight line of distinctly lesser slope. This changing slope expresses in part the neutralization postulated by Weiss and Rapport[4] of the specific dynamic action of amino acids when these were added to relatively large quantities of protein. Weiss and Rapport compared the specific dynamic effects over the first

four hours of mixtures of gelatin and casein, and an amino acid, glycine or alanine, with the specific dynamic effects, over the same period of time of the same quantity of each of the substances when administered separately. The results observed were that the increase in metabolism was distinctly less than the linear sum of the effects of the protein and of the amino acid when administered separately. Weiss and Rapport concluded from these results that the specific dynamic effects of the amino acids were neutralized when administered with protein.

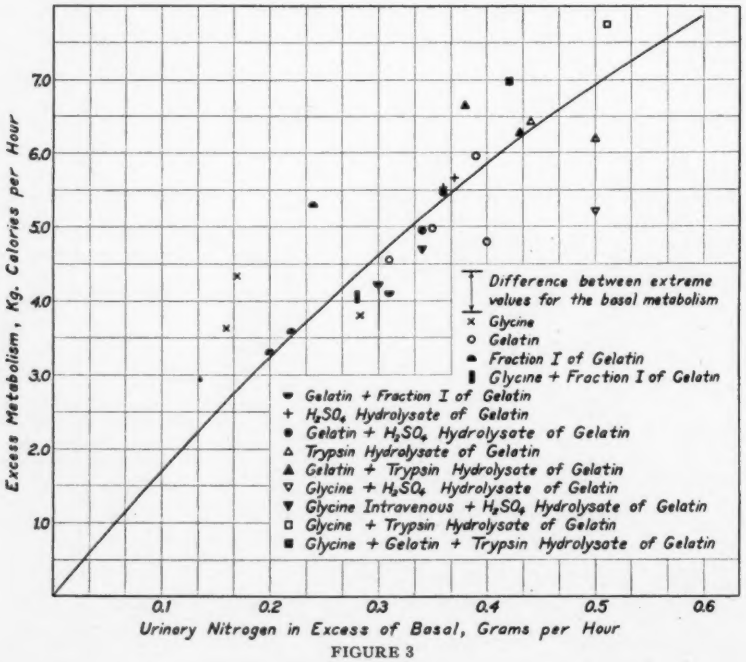

FIGURE 3

Figures 1, 2, and 3 show that an apparent neutralization of the specific dynamic action was obtained during the first four hours, whenever the quantity of nitrogen metabolized in this period was so large that it fell on the portion of the curve which does not pass through the origin when extended. This begins to be indicated when rather large quantities of amino acids were given, for which no neutralization phenomenon was postulated, and is clearly shown also when increasing quantities of beef were administered.

The data of Weiss and Rapport show an increase in metabolism, after the ingestion of 1.87 gm. of glycine, of 21.4 per cent; when an additional

1.87 gm. of nitrogen in the form of glycine were given, the increase over the basal was 30.9 per cent; and when 1.55 gm. of nitrogen in the form of gelatin were given with 1.87 gm. of glycine nitrogen, the increase in metabolism over the basal was 26.5 per cent. The increase in metabolism for this amount of gelatin alone was 12 per cent. Therefore, when the bolus ingested was increased by additional glycine or gelatin, the effect obtained was less than a linear sum of the separate specific dynamic effects; and the effect of the glycine-gelatin mixture was substantially the same as when the same amount of nitrogen was given in the form of glycine. Here, therefore, glycine exerted a neutralizing effect upon itself, similar to that

TABLE 1—(See Figure 1)

THE RELATION BETWEEN THE EXCESS NITROGEN EXCRETED AND THE SPECIFIC DYNAMIC ACTION OF PROTEIN (3) (4)

SUBSTANCES INGESTED	NITROGEN EXCRETION IN EXCESS OF BASAL PER HOUR	METABOLISM INCREASE OVER BASAL, PER HOUR	METABOLISM INCREASE OVER BASAL
	GM.	CALORIES	PER CENT
Beef	0.15	4.99	30.3
	0.29	5.86	35.6
	0.31	5.86	35.6
	0.37	4.92	29.8
	0.64	6.27	38.1
	0.89	8.43	51.2
	1.18	9.81	59.5
Casein	0.17	4.93	29.9
	0.19	5.18	31.4
Casein + HCl	0.19	4.65	28.2
Beef + Casein	0.21	5.29	32.1
Gliadin	0.27	5.76	35.0
	0.29	5.88	35.7
Codfish	0.34	5.73	34.8
Chicken	0.23	5.11	31.0

postulated when larger amounts of gelatin were given with glycine. When the amounts of nitrogen ingested and metabolized in the first four hours were larger, the neutralization was more marked.

A second factor in the phenomenon of neutralization is that the relationship between nitrogen ingested and nitrogen metabolized, i.e., excreted, is similarly not represented by a straight line, but by a curve of diminishing slope. For instance, the excess nitrogen excreted during the four-hour period after the ingestion of 10 gm. of glycine was 0.21 gm. per hour; after 20 gm. of glycine, 0.36 gm.; after 10 gm. of gelatin the excess urinary nitrogen was 0.15 gm.; after 40 gm. of gelatin, 32 gm.

When the exciting substances are similar, though not necessarily identical, the same increase in metabolism was obtained when the amount of nitrogen metabolized was the same. This is shown in the following

experiments of Weiss and Rapport. When 6 gm. of nitrogen were administered in the form of gelatin, there was an increase in the urinary nitrogen of 0.30 gm., and a 29.3 per cent increase in metabolism. In another experiment, when 6 gm. of nitrogen in the form of gelatin and 1.87 gm. of nitrogen as glycine were given together, the increased urinary nitrogen was 0.29 gm. and the increase in metabolism, 28.2 per cent. The apparent neutralization in this case of the specific dynamic effect of glycine is obviously due to some slowing of the rate of metabolism of

TABLE 2—(See Figure 2)

THE RELATION BETWEEN THE EXCESS NITROGEN EXCRETED AND THE SPECIFIC DYNAMIC ACTION OF ALANINE, GLYCINE, AND GELATIN (3) (4)

SUBSTANCES INGESTED	NITROGEN EXCRETION IN EXCESS OF BASAL PER HOUR GM.	METABOLISM INCREASE OVER BASAL PER HOUR CALORIES	METABOLISM INCREASE OVER BASAL PER CENT
Gelatin + Glycine + Alanine	0.33	5.58	33.9
Gelatin	0.13	2.09	12.7
	0.17	1.88	11.4
	0.27	4.71	28.6
	0.28	4.93	29.9
	0.30	4.53	27.5
	0.35	5.14	31.2
Glycine	0.20	3.51	21.4
	0.20	3.57	21.7
	0.23	3.50	21.2
	0.36	5.09	30.9
Alanine	0.21	3.21	19.5
	0.29	3.76	22.8
Glycine + Alanine	0.40	5.19	31.5
	0.45	5.46	33.1
Gelatin + Glycine	0.29	4.64	28.2
	0.33	4.36	26.5
	0.48	6.03	36.5
Glycine (subcutaneous)	0.16	2.48	15.1
Glycine (intravenous)	0.21	3.07	18.6
Gelatin + Glycine (intravenous)	0.39	5.01	30.4
Gelatin + Glycine (subcutaneous)	0.30	5.07	30.8
Gelatin + Alanine	0.35	4.98	30.2

the mixture, and not to a neutralization of the specific dynamic effect of glycine, *per se*. When large amounts of protein or amino acids or mixtures of the two are ingested there is a superposition of two non-linear effects; one is that the relationship between nitrogen ingested and nitrogen metabolized is not a linear function; the other, that the relationship between nitrogen metabolized and the increase in energy metabolism, over the first four hours, is also less than linear.

The term neutralization for this phenomenon implies that the specific

dynamic effects of these amino acids would be absent even when all the nitrogen ingested was accounted for in the urine. In the experiment in which glycine was administered with casein, during the first four-hour period of the experiment, only 14 per cent of the ingested nitrogen was excreted; in the case of casein and alanine, 12 per cent; gelatin and alanine,

TABLE 3—(See Figure 3)

THE RELATION BETWEEN THE EXCESS NITROGEN EXCRETED AND THE SPECIFIC DYNAMIC ACTION OF GLYCINE, GELATIN AND GELATIN HYDROLYSATES (5) (8)

SUBSTANCES INGESTED	NITROGEN EXCRETION IN EXCESS OF BASAL PER HOUR GM.	METABOLISM INCREASE OVER BASAL PER HOUR CALORIES	METABOLISM INCREASE OVER BASAL PER CENT
Glycine	0.16	3.62	29.2
	0.17	4.33	34.9
	0.28	3.79	27.7
Gelatin	0.31	4.56	36.8
	0.35	4.98	40.2
	0.39	5.96	43.6
	0.40	4.81	35.2
Fraction I of Gelatin	0.20	3.31	26.7
	0.22	3.60	29.0
	0.24	5.31	42.8
Glycine + Fraction I of Gelatin	0.28	4.06	32.7
Gelatin + Fraction I of Gelatin	0.30	4.21	34.0
	0.31	4.08	32.9
H_2SO_4 Hydrolysate of Gelatin	0.36	5.53	40.5
	0.37	5.67	41.6
Gelatin + H_2SO_4 Hydrolysate of Gelatin	0.34	4.96	36.3
	0.36	5.48	40.1
Trypsin Hydrolysate of Gelatin	0.44	6.42	47.0
	0.50	6.20	45.4
Gelatin + Trypsin Hydrolysate of Gelatin	0.38	6.64	48.6
	0.43	6.27	45.9
Glycine + H_2SO_4 Hydrolysate of Gelatin	0.50	5.22	38.2
Glycine (intravenous) + H_2SO_4 Hydrolysate of Gelatin	0.34	4.71	34.5
Glycine + Trypsin Hydrolysate of Gelatin	0.51	7.75	56.7
Glycine + Gelatin + Trypsin Hydrolysate of Gelatin	0.42	6.98	51.1

16 per cent; gelatin and glycine, 10 gm. of each, 39 per cent; and of a mixture of gelatin, glycine and alanine, 13 per cent. The effects observed obviously relate only to rates of absorption and metabolism; the data do not permit of any conclusions regarding absolute effects.

If the observations during the first four hours are confined to quantities

of nitrogen not more than 0.3 or 0.4 gm. per hour greater than the basal, the curvature is such that there is not a wide departure from a linear relationship between excess metabolism and nitrogen excreted in excess of the basal. For instance, the results in tables 2 and 3 depicted in figures 2 and 3 yield a relatively constant value of approximately 16 kg. calories per gram of nitrogen excreted in excess of the basal. The results of Wilhelmj and Bollman[6] confirm this conclusion. These authors concluded from their observations on the effects on metabolism of injected glycine, alanine and phenylalanine that "The relationship between the specific dynamic action may be expressed in at least six different ways, and reasons are given for believing that the most suitable manner of expressing this relationship is as calories and extra heat produced by each millimol of the amino acid deaminized." In a later paper Wilhelmj and Mann[7] present further experimental evidence in support of this correlation.

The results plotted in figures 1, 2 and 3 are the individual observations, when these are given; the alignment of points is distinctly better if averages only are used; the widest divergence from the mean position of the points is usually little greater than the difference between the extreme values for the basal metabolism.

It is interesting that the slope of the straight line portion of figure 1 yields a value for the increase in metabolism per gram of excess urinary nitrogen of 5 kg. calories, which is of the same order of magnitude as the energy consumption of the kidney observed in man, per gram of urinary nitrogen. This coincidence suggests the following possible interpretation of the results given above. The specific dynamic action of protein or amino acids, at any given time, is an expression of the resultant of two rates, of which one is the excretion of nitrogen, and the other, the metabolism and probably also the formation of the deaminized residues. The first steep part of figure 1, which corresponds to nearly the whole of figures 2 and 3, represents these combined effects. The flattening of the curve is due to the attainment of a maximum rate of deamination and metabolism of the deaminized fragments. Once this maximum intensity of metabolism is attained, this factor no longer contributes to the slope of the curve. The continuing straight line portion expresses the metabolism of the kidney. The persistence of the slope unchanged indicates that the rate of urea excretion can be increased by the ingestion of increasing amounts of protein, long after the maximum rate of metabolism of the deaminized residues has been attained; and that the excretion of nitrogen lags behind deamination.

The hypothesis advanced here is based chiefly on observations made over the first four hours following the ingestion of proteins or amino acids. The fact that even in this short period processes with different rates are discernible indicates that if a sufficiently large number of observations

over a longer period were available, further complications would be observed, due, for example, to the completion of some processes and the continuance of others.

Further evidence of the association of the specific dynamic action of protein with nitrogen metabolism is contained in the observations of Wishart[9] on the variations in the basal metabolism with changes in the nitrogen excretion, resulting from the variations in the daily protein intake. During the period of observation the daily nitrogen excretion varied in one subject from 5.3 gm. to 23.2 gm., and in another from 3.8 gm. to 16.5 gm. The results obtained could be expressed by the equations $x = 28.456 + 0.449y$ in the one case, and in the other by $x = 28.269 + 0.448y$; where x is the basal metabolism in calories per square meter per hour, and y the urinary total nitrogen in g. per day. The correlation coefficients were, $+0.84 \pm 0.036$, 0.70 ± 0.05. These results show a linear relationship between nitrogen excretion and increase in metabolism. The coefficient of y is of the same order of magnitude as that for the first part of figure 1, or for figures 2 and 3, which includes the effects of the metabolism of nitrogen and the deaminized fragments. This is in accord with the results obtained on dogs because even the largest amount of nitrogen metabolized here corresponds to points on the lower portions of figure 1, or figures 2 and 3. For instance, 23.2 gm. of nitrogen per day would correspond in a 20-kg. dog to about 0.3 gm. per hour. As pointed out above, in this range of excess nitrogen excreted the specific dynamic effects observed on the dog are defined approximately by a linear function of the amount of nitrogen excreted in excess of the basal.

The observations in cases of endocrine disturbance, also show the relationship between nitrogen metabolism and the specific dynamic action of protein. Liebeschütz-Plaut[10] noted a reduction in the specific dynamic action of protein in cases of *dystrophia adiposogenitalis*. This was confirmed by Liebesny[11] on men, and by Foster and Smith[12] on hypophysectomized rats. Gaebler[13] in Lusk's laboratory compared the specific dynamic effects of meat, in the dog, before and after hypophysectomy; and found before the operation an average increase in excretion of nitrogen per hour, after a meat meal, of 0.16 gm., and an increase in metabolism of 4.4 calories per hour; after the hypophysectomy the average increase in nitrogen excretion (over the normal basal value, which is probably too high in this case) was 0.09 gm. per hour, and the increase in metabolism 3.2 calories per hour. Gaebler's conclusion was that "discrepancies in the amount of protein catabolized probably account for the difference in the extent to which the heat production was increased in the two animals." Lusk's summary of the findings in this condition,[24] shows that the general rate of metabolism is lowered after hypophysectomy or pituitary deficiency. Foster and Smith found that the metabolism in these animals

may be restored to normal by anterior pituitary homotransplants, or by daily injections of thyroid extract.

Taken in conjunction with Gaebler's experiments, the findings in case of hypophyseal deficiency indicate no real reduction in the specific dynamic action, but rather a slowing of the rate of metabolism. Per gram of nitrogen metabolized, the specific dynamic action is the same as in the normal. Lauter[14] showed that in the case of obese individuals the specific dynamic action of protein is less, during the first two hours, than in normal individuals, but is the same as in the normal over a period of six hours.

The correctness of the association of the extent of the specific dynamic action with the nitrogen metabolism, i.e., with the increase in urinary nitrogen, is consistent with the time relationships. Lusk reproduces in "The Science of Nutrition" a curve from the data obtained by Aub and Dubois on human beings, which shows the concomitant increases in urinary nitrogen, urinary sulfate, and in the rate of metabolism. Another similar curve is reproduced from the data of Williams, Riche and Lusk. It may be added that, following the ingestion of some proteins, there is an increase also in urinary phosphate. The excretion of sulfate and of phosphate imposes work upon the kidney, in much the same manner, though to a lesser extent in the case of protein, as in the excretion of urea; and therefore, also figures in the specific dynamic action of protein.

The observations of Wilhelmj, Bollmann and Mann[15] that injected amino acids do not exert any specific dynamic effect in hepatectomized dogs, definitely eliminates the view that the specific dynamic action of amino acids is due to a general stimulation of the cells of the body by the amino acids *per se*. The earlier observations of Rapport and Katz[16] of an increase in metabolism following the addition of glycine to the circulating blood of an isolated hind leg preparation is open to the criticism that the concentration of glycine in the circulating blood was relatively enormous, about one per cent. The difficulty of interpreting this observation of Rapport and Katz is further increased by the recently reported observations of Needham[17] that neither glycine nor alanine increased the respiration of a minced muscle preparation, though a marked increase was observed with glutamic acid.

The observations of Grafe[18] that ammonium chloride and acetamide exert a large specific dynamic effect, with the failure of the amino acids to provoke an increase in metabolism in the hepatectomized animal, and the general correlation of the increase in metabolism with the increase in urinary nitrogen, indicate that either the whole or part of the composite process of urea formation and excretion are responsible for a large fraction of the specific dynamic action of protein.

The view that the specific dynamic action of protein is due to the

formation of glucose from the deaminized fragments chiefly of glycine and alanine is again put forward by Chambers and Lusk in a recent paper.[22] This proposal meets with the difficulty of accounting for the important observation of Rapport[3] that the specific dynamic action of a variety of proteins is nearly the same, and was lowest among the proteins tried for gelatin, which contains the most glycine and alanine, and was highest for gliadin which contains over 40 per cent glutamic acid and very small amounts of glycine and alanine.

In this paper, Chambers and Lusk reassert the contention that no specific dynamic action is exerted by glutamic acid. Grafe, Rapport and Beard, and Terroine and his collaborators[19,20] have reported that glutamic acid exerts a considerable specific dynamic action. Johnston and Lewis[23] observed that glutamic acid, administered to rabbits *per os*, was absorbed very slowly compared to alanine and glycine. On the other hand, injected intravenously glutamic acid was deaminized as quickly as alanine or glycine. Rapport and Beard, from their own observations, suggested that the apparently low specific dynamic effect of glutamic acid is due to the slowness of its absorption. The data in the protocols of Chambers and Lusk suggest that this discrepancy may be resolved by considering the amount of glutamic acid deaminized during the period of observation. In one experiment, the data of Chambers and Lusk show that in the third hour after the ingestion of 20 gm. glutamic acid by a normal dog, with an increase in urinary nitrogen of only 0.03 gm. per hour, the metabolic rate was 1.8 per cent lower than the average basal value. In three other experiments, when the same amount of glutamic acid was given with 10 gm. lard (the specific dynamic action of which is only 3.5 per cent and which alone induced no increased excretion of nitrogen), during the second and third hours there was an increase in the hourly urinary nitrogen of 0.1 gm., accompanied by an increase in the metabolic rate of 15 per cent. The difference between the rate of absorption and metabolism of glutamic acid as compared with other amino acids is illustrated in the effect on the same dog of 300 gm. lean meat. The hourly urinary nitrogen was increased eight-fold, and the metabolic rate 52.3 per cent.

After these observations were made the animal was phlorhizinized and glutamic acid again administered; from the results obtained, Chambers and Lusk concluded that there was no specific dynamic action of this amino acid in the phlorhizinized animal. The values of the basal metabolism, 26.5 calories per hour, with which the glutamic acid figures were compared, is an average of the following values obtained on the days of the experiments: 23.4, 29.4 and 26.7. It is not possible from the urinary nitrogen figures, which were also irregular here, to obtain a reliable estimate of the amount of glutamic acid deaminized, except that it was small.

The strongest evidence upon which the theory suggested by Lusk is

based, that the specific dynamic action is due to the conversion of the deaminized residues of the amino acids into glucose, is derived from the observations that the specific dynamic action of glycine and alanine is the same in the phlorhizinized dog, as in the normal animal. In the interpretation of this result, Lusk assumes that the phlorhizinized dog is unable to burn glucose, and that the glycine and alanine ingested are converted quantitatively into glucose. In Lusk's summary[24] of the effects of phlorhizin, the observations of Richardson and Shorr are quoted that the excised tissues of phlorhizinized rats burn glucose as freely as the normal; and further, that a variable quantity of ingested glucose is oxidized by the phlorhizinized animal. As Terroine and Bonnet[25] have pointed out in their discussion of this problem, even in the normal animal glucose probably is not oxidized as such, but in the form of some degradation product such as lactic acid. An alternative and equally tenable interpretation, therefore, of the observations on the phlorhizinized dog is that the deaminized fragments of the amino acids are oxidized by the phlorhizinized dog in place of degradation products derived from glucose, with the result that an amount of glucose is spared from combustion, and is thereby excreted in the urine, which is approximately equivalent to the ingested amino acids. This would tend to lessen the total metabolic rate in the phlorhizinized as compared with a normal animal. On the other hand this is compensated for by the increase in the renal work, and hence the total metabolism, by the excretion of an increased amount of glucose.[1] The magnitude of this increase will be of the same order as the energy consumed by the kidney in the excretion of the nitrogen.

In their paper, Chambers and Lusk return to a discussion of the energy changes in the hypothetical conversion of glycine, alanine and glutamic acid into glucose and urea. The following equations are written:

for alanine: $2C_3H_7NO_2 + H_2O + CO_2 \longrightarrow C_6H_{12}O_6 + CON_2H_4$;

for glycine: $3C_2H_5NO_2 + {}^3/_2CO_2 + {}^3/_2H_2O \longrightarrow C_6H_{12}O_6 + {}^3/_2O_2 + {}^3/_2CON_2H_4$;

and for glutamic acid:

$$2C_5H_9NO_4 + 3O_2 \longrightarrow C_6H_{12}O_6 + H_2O + CON_2H_4 + 3CO_2.$$

The calculations of Adams[26] are referred to, who, with the aid of the Third Law of Thermodynamics, estimated that approximately 60,000 calories would be required to effect the change of state indicated above for alanine. The specific heat data at present available are too scanty to permit of reliable estimates of the entropies of alanine and of glycine. Accordingly, calculations of the free energy change incurred in the conversion of glycine to glucose and urea, although the standard free energies of glucose and urea are now known, are at present uncertain. An estimate

of the direction and magnitude of the energy change can be obtained by means of the First Law of Thermodynamics. From the data in the International Critical Tables, the following values have been calculated for the molar heats of formation: water, −68,000 calories; carbon dioxide, −94,000 calories; glucose (solid), −303,000 calories; urea (solid) −79,000 calories; glycine (solid), −126,000 calories; alanine (solid) −134,000 calories; and glutamic acid (solid) −237,000 calories. The value of $-\Delta H$ in the reaction postulated above for alanine is −24,000 calories per mol; for glycine, −66,000 calories; and for glutamic acid +129,000 calories. The values for the alanine and glycine reactions are nearly the same as those given by Adams.

Assuming for a moment the reality of these reactions, it may be concluded from the above values of $-\Delta H$, that the conversion of glycine and alanine into glucose and urea are endothermic processes requiring the driving force of another exothermic process. And if it is assumed that the relationship is not perfectly reversible between the endothermic and exothermic processes, the specific dynamic action will be the difference between the amount of energy produced and the amount used in the endothermic reaction. There are no data for estimating the degree of reversibility of such a reaction in the body. If a 25 per cent efficiency is assumed, then the specific dynamic action per mol of glycine would be approximately 200,000 calories. This is the order of magnitude of the specific dynamic action of glycine. But on the same basis, the specific dynamic action of alanine should be one-third that of glycine. The experiments of Lusk, and of Wilhelmj and Bollman, and of Wilhelmj and Mann are in accord that the specific dynamic action of alanine per mol of amino acid deaminized is practically the same as that of glycine. $-\Delta H$ for glutamic acid is positive; the reaction is, therefore, exothermic. The value of $-\Delta H$, moreover, is quite large, equivalent to the heat of combustion of $^1/_5$ mol glucose. There is no obvious justification for assuming, as do Lusk, and Aubel,[30] that because this reaction is itself exothermic, that it is not responsible for any specific dynamic action. Rather the reverse, it seems that this is the one reaction of the three which would certainly exert a specific dynamic effect. Even in the cases postulated for glycine and alanine the specific dynamic action is due to exothermic processes. As pointed out above, gliadin, which contains over 40 per cent of glutamic acid, exerts a large specific dynamic effect. In this connection reference may be made to the observation of Terroine and Bonnet that the whole of the heat of combustion of ingested alcohol can be recovered as heat evolved by the organism.[27]

Furthermore, the values of $-\Delta H$ calculated for the above reactions include the energy changes for deamination and urea formation. There is no *a priori* reason for considering that the latter reactions and the excre-

tion of urea are negligible in the energy balance. The failure of sodium lactate and sodium glycollate to exert a large specific dynamic action[21] might be taken to indicate that the formation of glucose from the deaminized residues of the amino acids is a negligible element. In the case of glycine and alanine, also, the mechanism postulated for the formation of urea (the dehydration of ammonium carbamate) has been severely criticized on chemical grounds by Werner.[28] If urea normally appears by way of cyanic acid from the concomitant oxidation of the amino acids and glucose,[29] the metabolism of glycine and alanine probably are exothermic instead of endothermic processes. This criticism of the view of Lusk and Aubel[30] regarding the cause of the specific dynamic action of protein has many points in common with the views of Terroine and Bonnet.

The values obtained for the specific dynamic action of the amino acids show that the work of the kidney in eliminating nitrogen cannot be responsible for the whole of the increase in metabolism. It was shown that the excretion of nitrogen in man incurs an expenditure of 6–11 kg. calories per gram of nitrogen.[1] In experiments in which most of the injected amino acid was accounted for, Wilhelmj and Bollman obtained in dogs the following values, expressed as kg. calories per gram of amino acid deaminized: for glycine, 35, 29 and 21; for alanine, 22, 28 and 20; and for phenylalanine, 56 and 44. In the later experiments of Wilhelmj and Mann, the values obtained for glycine were 16 and 18; and for alanine, 16 and 19. The few observations on the effect of tyrosine suggest that the specific dynamic action of this amino acid, per gram of nitrogen metabolized, is greater than that of phenylalanine. The data of Rapport, Weiss and Rapport, and of Rapport and Beard give somewhat low values for glycine, in accord with the lower values for glycine and alanine obtained by Wilhelmj and Mann. The values for glutamic acid obtained by Rapport and Beard are nearly as great as those of phenylalanine; but in these experiments only a small fraction of the ingested amino acid was accounted for.

If the specific dynamic action of protein were due only to the energy released in the deamination, formation, and excretion of urea, the specific dynamic action would be practically the same per mol for all the monoamino acids. Terroine and Bonnet found in the frog that, per gram of nitrogen injected, the specific dynamic action of glycine, alanine, aspartic acid, glutamic acid, valine, leucine, cystine and lysine was approximately 8.4 calories; for tyrosine and phenylalanine, 9.2 calories; and for tryptophane and histidine, 10 calories.[31] It is interesting and possibly significant, in view of these low values, that in the urine of the frog the concentration of nitrogen is practically the same as the non-protein nitrogen of the blood,[32] so that no additional work is imposed upon the kidney by the ingestion of proteins or amino acids, and hence in the frog the

specific dynamic action of amino acids might be predicted to be 6–11 kg. calories per gram of nitrogen less than in the mammal. Terroine and Bonnet[25] observed similar values in the rabbit. But the values obtained by the French observers on the rabbit may be too low if they are taken to signify the specific dynamic action per gram of nitrogen excreted in the dog or man. For instance, 16 gm. of glycocoll were injected into a rabbit weighing 2.7 kilos. The observations were made over a period of six hours, and it is improbable that in this time such a relatively very large quantity of the nitrogen was completely metabolized and excreted. On the other hand the energy consumption during renal excretion in the rabbit is probably less than in the dog or man.[1] The evidence favors acceptance of the higher values, in the case of the higher animals, obtained by Lusk, Weiss and Rapport and by Wilhelmj and Mann.

It is possible that the difference in the specific dynamic effects of different monoamino acids may be due to the metabolism of the varying number of carbon atoms which they contain. This would account for the greater specific dynamic effect, per mol metabolized, of phenylalanine, tyrosine and glutamic acid. But the data available are too incomplete for analysis.

An estimate of the value of the specific dynamic action of protein, as distinguished from that of an amino acid, can be derived from the observations of Wishart on man, referred to above. The equations of Wishart expressed as calories per hour, instead of calories per square meter per hour, are:

$$x = 52.65 + 0.83y,$$

and

$$x = 48.0 + 0.76y.$$

Taking y as gm. of nitrogen excreted per hour, the equations become:

$$x = 52.65 + 20y,$$

and

$$x = 48.0 + 18y.$$

The coefficients of y, 20 and 18, represent the increase in metabolism per hour per gram of nitrogen excreted. It is confirmatory of the explanation of the specific dynamic action of protein proposed here, that these values for the increase in metabolism induced by protein, calculated from changes in the basal rate with changes in the nitrogen excretion, are of the same order of magnitude as those calculated from the directly observed specific dynamic action of protein. These figures carry additional weight because the errors due to long periods of observation or to incomplete metabolism of the ingested protein could not occur here.

The results obtained indicate that the additional work imposed upon the kidney is responsible for 25–60 per cent of the specific dynamic action of protein and amino acids. The remainder is due to the "specific dynamic action proper" resulting from the metabolism of the nitrogen and of the

carbon. In this estimate it is assumed that the energy consumed by the kidney in the excretion of nitrogen is approximately the same in the dog as in man, in spite of the lower values of the dog obtained by Steck.[2] No details are given in the quotation of Zuntz regarding Steck's experiments; while the oxygen consumption values for the dog's kidney during urea diuresis indicate that the work of the kidney in the excretion of nitrogen is approximately the same in the dog as in man.[1]

The values obtained for the increased energy consumption by the kidney in man, per gram of nitrogen excreted indicate that increased renal function may be responsible for as much as 40 per cent of the specific dynamic action of protein observed in man by Aub and Dubois.

Summary.—Taken in conjunction with the data on the efficiency of the process of renal excretion, analyses of the data bearing upon the relation of the increase in metabolism following the ingestion of protein or amino acids reveals:

(1) A close correlation between the specific dynamic action of proteins or of amino acids and the increase, over the basal level of excretion, in the urinary nitrogen;

(2) That neither the direct experimental evidence nor the considerations of the energy relations support the view that the specific dynamic action of protein is necessarily due to the conversion of the deaminized fractions into glucose.

(3) The values of the specific dynamic action of amino acids and of proteins indicate that 25–60 per cent is due to the work imposed upon the kidney. The remainder of the specific dynamic action of protein or amino acids is due to the metabolism of the nitrogen and the carbon, though it is not possible, from the evidence, to estimate the proportion for which each is responsible.

(4) This hypothesis, that the specific dynamic action of protein is due to at least two distinct processes, which do not proceed at similar rates, provides an explanation for some hitherto anomalous phenomena in the specific dynamic action of protein.

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THE CALIFORNIA EARTHQUAKES OF NOV. 28, 1929, AND THE SURFACE LAYERS OF THE EARTH IN CALIFORNIA

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On November 28, 1929, at about 11h. 49m. A.M., Pacific Standard Time and again about 3 minutes later, the west central section of California and the adjoining portion of Nevada were shaken by earthquakes which attained an intensity of about VII, Rossi-Forel scale, about 5 miles southeast of Aberdeen, California. Here the earthquakes cracked concrete reservoirs, broke plates on shelves and overturned an alarm clock. At the head of Goodale Creek, some 20 miles southwest of Bishop, a large landslide occurred at the time of the earthquakes. According to one report, "considerable of south side of mountain top slid down into gorge on the headwater of Goodale Creek about $1^1/_2$ miles north of Division Creek Power House." This landslide was photographed from an airplane while it was taking place. The photograph is in the hands of Prof. J. C. Jones

of the University of Nevada. At Bishop and Big Pine the intensity was probably about VI. The isoseismals of these earthquakes are shown in figure 1. The area of greatest intensity cannot be accurately bounded since information from the Sierran region is sparse. Therefore only the eastern boundary of this region is drawn. The peculiar shape of the isoseismals resembles that of the isoseismals of other earthquakes in this region. It is doubtless due to the nature of the geologic foundation of the region. In the region marked III–IV the earthquake intensity was characterized by rattling of doors and windows and creaking of houses. In the district marked I–II the earthquake was felt but the phenomena mentioned in the characterization of III–IV were not observed. The outer boundary of the region I–II marks the limits of the region in which the earthquake was felt.

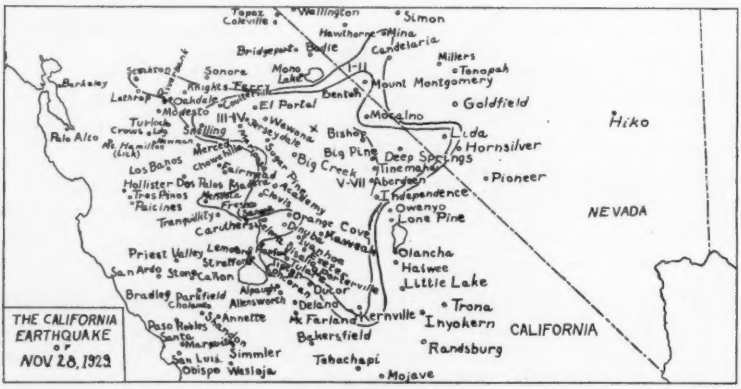

FIGURE 1

Available for the instrumental study of this earthquake were seismograms from stations at Haiwee, Tinemaha, Santa Barbara, Mt. Wilson, Pasadena, La Jolla (all loaned by Mr. H. O. Wood), Reno (loaned by Prof. J. C. Jones), Tucson (loaned by the Director, U. S. C. and G. S.), Santa Clara (loaned by Father Henry), Mt. Hamilton and Berkeley.

These were the first American earthquakes to be so well recorded by so many stations relatively close to the epicenter. As such the records merited close study since another earthquake so well located may not occur for some time.

The location of the epicenter satisfactorily was difficult. It has been found in European earthquakes that it is possible to draw travel-time curves of the various pulses recorded that are straight lines. Therefore, in this case an epicenter which would allow such interpretation was considered good. The first motion observed in an earthquake is usually

longitudinal in character. Therefore it is possible to locate an epicenter by directions from stations in case the first wave is well recorded and the exact orientation of the instruments is known. In the present case this method was used with the three near-by stations: Tinemaha, Haiwee and Lick Observatory. These stations were all equipped with Wood-Anderson seismographs. With so small a mass it has proved difficult to be certain of the exact orientation and therefore there is doubt within a few degrees as to the component measured.

In the location of the epicenter here adopted and indicated on figure 1 as a cross the following criteria were used: (1) it is in the exact direction

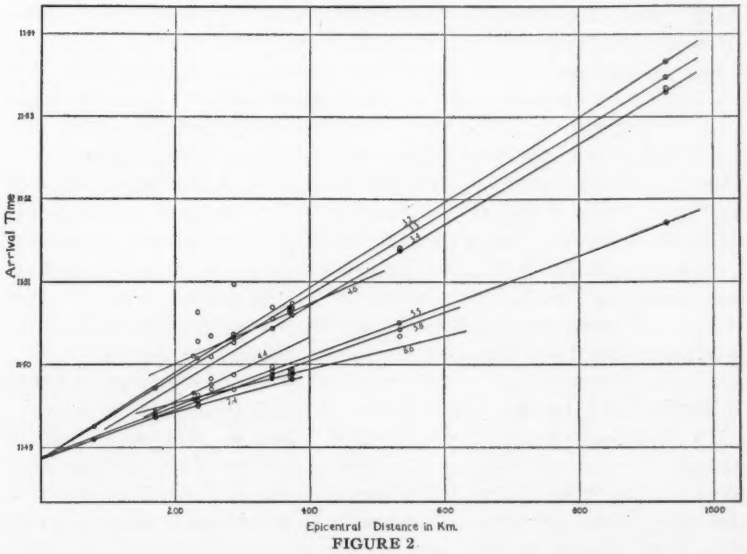

FIGURE 2

from Tinemaha, the nearest station, that was computed from first motion; (2) it lies midway between the points where the computed direction lines from Haiwee and Lick Observatory cut the Tinemaha direction line; (3) it allows travel-time curves to be drawn as straight lines in reasonable accord with those observed in Europe. The speed of $\overline{P}$ checks with that found by Wood for California explosions.

The coördinates of the epicenter are 37°31′ north latitude, 119°02′ west longitude.

The times of arrivals at Tinemaha and Haiwee as well as the direction from Tinemaha excluded all possibility of the source of the earthquakes having been at the location of the landslide mentioned above. The

landslide was apparently a secondary phenomena caused by the earthquake.

The times of arrival of the various wave groups at each station were plotted against the epicentral distance of that station. It was found that almost all of these pulses might be accounted for by a series of straight line travel-time curves. These are given in figure 2. On some of the lines are only three points. If such a type of interpretation had not been found satisfactory in Europe, its use here would have been more questionable. In no case was a line drawn when only two points on it were observed. In the case of the first arrival at Haiwee it appeared that two waves were arriving together and in plotting only one point could be put down. Each straight line is marked with a number which gives the speed in kilometers per second indicated by the reciprocal of its slope.

My interpretation of the curves is as follows: The wave showing speed 5.8 km./sec. is the direct P (compression—rarefaction wave) from the focus to the station. The wave of speed 5.5 km./sec. is the P wave which had its origin in the incidence of the primal S wave at the surface. This is the wave observed by Jeffreys[1] in his study of English earthquakes, and theoretically explained by Nakano.[2] Jeffreys pointed out that since the strain released at the source may be expected to be largely shear, the P wave originating at the source may be expected to be small, and for his earthquakes he did not observe it. In the present case the direct wave is not observed at a distance while the indirect P is so observed. The travel-time line of the direct P intersects the travel-time of the direct S (speed, 3.3 km./sec. at the epicentral distance of zero giving the time of occurrence, 11h. 48m. 52s).

The identification of both the direct P, which we shall call $\overline{P}$, and the indirect P, which we shall call $\overline{P}_s$, allows a new and exact method of determining the depth of focus. The time intercept of $\overline{P}_s$ lies 1 second above that of $\overline{P}$. Now if $\overline{P}_s$ has its origin in the incidence of the primal S at the surface the angle of incidence of S at which the transformation takes place is $\theta = \sin^{-1} 3.3/5.8$. The difference in the intercepts of $\overline{P}$ and $\overline{P}_s$ is then very approximately equal to $h(\sec \theta/3.3 - \tan \theta/5.8) = 1$, where h is the depth of focus. The value of h is then about 4 or 5 km. Such a depth might be considered rather inconsistent with the differences in speed of $\overline{P}$ and $\overline{P}_s$.

The wave of speed 7.4 km./sec. is interpreted as the P wave which has penetrated into the second layer and then emerged P^*, and the wave of speed 8.6 km./sec. as P_n, the wave which has penetrated the medium underlying the second layer. The wave of speed 4.6 km./sec. is S_n the shear wave corresponding to P_n. S^* was not observed. Its computed speed $7.4/1.8 = 4.1$ checks the value of 4.0 which I have concluded for short Love waves under the Pacific.[3]

Knowing the speeds of $\overline{P}$ and P^*, the intersections of their two travel-time curves and the depth of focus, the thicknesses of the two surface layers of the earth may be computed.

If we assume after Jeffreys, as we did above, that the P energy was mainly derived by reflection from the surface, the path of P^* should be taken as S to the surface, reflected as P and refracted along the boundary as P and again out as P. Taking such a path we find the thickness of the outer or granite layer to be about 23 km. If P^* is considered as P throughout path from the focus, then the computed thickness is about 30 km. The former hypothesis is more consistent with the interpretation which led to fixing the depth of focus.

The curve drawn for the travel time of P_n is perhaps more questionable than that for P^* since it at no point is the first wave observed. However, a corresponding S_n is observed. The intercepts of these two curves with those of $\overline{P}$ and $\overline{S}$, respectively, lead to a computed thickness of the intermediate layer of 80 to 90 km.

From isostatic considerations this value seems entirely too large if the values for the thickness of the surface layer under the Pacific as hitherto computed are correct.

For computation, take 1.5 km. for the height of the Sierra block and 4 km. for the depth of the Pacific. Let x be the thickness of the layer underlying the Pacific and let its density be 3.1. Let y be the thickness of the intermediate layer under the Sierras and its density 3.1. Let the density of the underlying medium in both cases be 3.3 and let the density of the granite be taken as 2.7, and the thickness of the granite layer 23 km.

Then for isostatic balance the following equation must be satisfied

$$3.1x + 4 + (17.5 + y - x)3.3 = 23{\cdot}2.7 + 3.1y$$
$$y - x = 1.7.$$

That is, the thickness of the intermediate layer under the Sierras must be very closely that of the upper layer under the Pacific as well as being of the same material. Angenheister[3] has found a value of 14 km. for the Pacific layer from dispersion of Rayleigh waves. I have found a value from dispersion of Love waves of 20 to 40 km. with the latter value the better in my opinion.[4] Gutenberg[5] prefers to explain dispersion under the Pacific differently than he explains it elsewhere, i.e., by a gradual change in constants with depth. It seems more consistent to me to apply the same type of reasoning for dispersion under the Pacific that is applied elsewhere.

Therefore, from isostatic considerations and the value of 40 km. which I found for the thickness of the Pacific layer I am inclined to prefer a value of some 40 km. for the intermediate layer under the Sierras to the 80–90 km. value which the local earthquake data have suggested. In such a

case it must be concluded that the earthquake was not of sufficient intensity to set up a P_n wave or a S_n wave large enough to be recorded at the stations and that the observed arrivals credited above to P_n and S_n were some other waves. It is true that the first wave was extremely weak at many stations.

The wave interpreted as $\overline{P}$ comes in first as a rarefaction at Tinemaha, Haiwee, Berkeley and La Jolla. It records first as a compression at Reno, Mount Wilson and Pasadena. At Santa Barbara it is almost transverse. This is a quadrant distribution and suggests elastic rebound along a fault bearing N 18° W ca., the west side moving south relative to the east side. The recent results of the United States Coast and Geodetic Survey infer such a general drift, i.e., they show a displacement south-westerly of triangulation stations in the Coast Range region, relative to Lola and Round Top in the Sierras.

Although I refer to these as rarefactions and compressions the motions are not longitudinal strictly. The following table gives the direction of first motion of $\overline{P}$ at the station and also the bearing of the epicenter at the station. The directions of motion are open to obvious sources of error. Of course for Tinemaha, Haiwee and Reno the two directions checked closely due to the method of locating the epicenter. The table gives also the angle between the direction of vibration of the wave and the path, and the number of the group in point of time of arrival:

STATION	MOTION	BEARING	ANGLE	NO.	NATURE
Reno	N45° W	S17° E	28°	2	Compr.
Berkeley	On E only	E7° S	7°	1	Raref.
Santa Barbara	S56° W	N10° E	46°	2	Raref.?
Pasadena	S45° E	N14° W	31°	3	Compr.
La Jolla	N53° W	N17° W	36°	2	Raref.

Similar information for $\overline{P}_s$ is given below:

STATION	MOTION	BEARING	ANGLE	NO. OF GROUP	NATURE
Haiwee	S53° E	N32° W	21°	2	Compr.
Santa Barbara	S49° E?	N10° E	59°	3	Transverse
La Jolla	S68° W	N17° W	51°	2	(Nearer transverse) compr.

In case of P^* we have

Reno	Largely to S As little E	S17° E	Small	1	Raref.
Pasadena	N	N14° W	14°	1	Raref.

In each case the phase of the first motion of direct $\overline{P}$ was opposite to that of the indirect. This indicates that reflection of S at the surface was the probable source of the indirect waves since refraction would not produce this change of phase.

However, it makes us pause when we note that the first wave at every station, without regard to what group we have assigned it, is a rarefaction.

The wave called P_n above recorded first at Lick, Santa Barbara and Pasadena as a rarefaction.

After the arrival of the P's, there is recorded at four stations a wave of velocity 4.4. The horizontal component of the direction of vibration of this wave is transverse to the path at Haiwee, Lick Observatory, Santa Clara and Berkeley. The following table shows this:

STATION	DIRECTION OF FIRST MOTION	BEARING OF EPICENTER	ANGLE
Haiwee	S55° W	N32° W	
Lick	N only	E3° N	87°
Santa Clara	N only	E3° N	87°
Berkeley	N45° E Down	E7° S	52°

A little later another wave of speed 3.4 km./sec. arrives. It seems more persistent as it is recorded at Tucson. The position of its travel-time line and the nature of the vibration suggests that it may be a Rayleigh wave, set up by the incidence of P^* at the surface. Its direction of vibration is given by:

STATION	DIRECTION OF FIRST MOTION	BEARING OF EPICENTER	ANGLE
Santa Barbara	N	N10° E	10°
Pasadena	E	N14° W	76°
La Jolla	S50° E	N17° W	33°
Tucson	S	N45′ W ca.	45°

However, the direction of vibration seems irregular.

After the waves called $\overline{S}$, S^*, S_n, arrives a group of velocity 3.2. Its horizontal component is transverse at Reno. At Berkeley it is registered as new only on the vertical.

In computing the direction of vibration in all the above cases trace amplitudes were used. For the Wood-Anderson instruments this is justifiable since the errors so introduced are of no greater order than those introduced by errors in orientation. For the Berkeley Bosch-Onori instruments it is all right since their magnification was closely the same. For Santa Clara and Reno the instruments were Wiecherts and the magnifications were not known.

In the tables giving readings of seismograms the various waves are correlated with $\overline{P}$, $\overline{P}_s$, P^*, P_n, $\overline{S}$, S_n, and in no case did a reading correlated with a symbol depart by more than two seconds from the travel-time line corresponding to that symbol; in most cases the discrepancy was less than a second.

The above discussion and the following tables apply to the first of the two earthquakes. The beginnings of the second earthquake were partially obscured by previous motion on a number of the records and the analysis of the records thus complicated. From the difference in first

arrival times at various stations it appeared that the epicenter of the second quake was closely that of the first. In cases where first arrivals did not indicate this, the beginning was very doubtful owing to previous motion. The first interval at Tinemaha was, however, much less than for the first quake, the wave of 3.4 (Fig. 1) apparently recording before $\overline{S}$.

Summary.—The epicenter of the earthquakes of Nov. 29, 1929, 11h. 48m. 52s. P. S. T., was fixed at 37°31′ north, 119°02′ west. The records were interpreted to indicate a depth of focus of about 5 km. and a thickness of the granitic layer in this region of between 20 and 25 km.

READINGS OF SEISMOGRAMS

Tinemaha

GROUP	TIME	(PERIOD) T	A (TRACE AMPLITUDE IN MM.)
$i\overline{P}_E$	11h.49m. 06.0s.	0.6s.	−7.4
$i\overline{P}_N$	06.0	0.4	+1.5
$i\overline{S}_E$	15.5	1.5	
S_N	15.5		

Haiwee

$i\overline{P}_N$	11h.49m. 22.3s.	0.2	+0.3
		1	+4
$i\overline{P}_E$	22.5	0.1	+0.1
		0.6	−3
$\overline{P}_{SE}$	23.8	0.4	+10
$\overline{P}_{SN}$	23.8	0.8	−7.5
i_E	26.7	0.6	−11
i_N	26.9	0.6	−7.7
$i\overline{S}_E$	43	0.7	+23
$i\overline{S}_N$	43	?	−15

Lick Observatory

$e\overline{P}_{SE}$	11h.49m. 35.0s.	0.7	+0.1
eP_{nN}	35.7	0.6	−0.1
eP_{nE}	36.0	0.7	+0.4
i_N?	39.5	0.5	+2
iS_{nN}	50 06.5	1	+5.5
iS_{nE}	07.0		−5

Santa Clara

i_E	11h.49m.42s.	3	<0.1
i_N	46		Blur
i_E	50	1	0.2
$\overline{S}_E$	50 06	2	0.3
$\overline{S}_N$	06	2	0.2
i_E	21	2	1.8
i_N	22	2	2

READINGS OF SEISMOGRAMS (*Continued*)

Reno

GROUP	TIME	(PERIOD) T	A (TRACE AMPLITUDE IN MM.)
eP^*_N	11h.49m.30s.	3	−0.1
eP^*_E	32	2	+<0.1
$i\overline{P}_E$	34	2	−0.2
$i\overline{P}_N$	34	2–3	+0.2
iP_{nN}?	38	2–3	+0.3
iP_{nE}?	39	2	+0.2
i_E	50 05	3	−0.5
i_N	17	2	+1
i_N	38	3	+3

Berkeley

$e\overline{P}_Z$	11h.49m.42s.	2	−0.1
$e\overline{P}_E$	42	2	+<0.1
i_z	53	2	−0.1
i_N	54	3	+0.1
i_E	54	3	+0.1
$i\overline{S}_E$	50 17		
iS_{nE}?	20	4	−0.1
iS_{nN}	20	2	+0.1
i_z	22	2	−2
e_E	58	7	−0.15
e_z	51 03	9	+0.1

Santa Barbara

eP_{nE}	11h.49m.50.3	0.3	+0.1
eP_{nN}	50.3	0.2	+0.1
$i\overline{P}_E$	51.8	0.3 & 0.6	−0.3
$i\overline{P}_N$	51.8	0.2	−0.2
$i\overline{P}_{nN}$	55.1	0.6	+0.6
$i\overline{P}_{nN}$	55.1	0.5	−0.7
i_E	58.7	0.5	+1.1
i_N	50 26.4	0.7	+2
iS_E	33.1	0.7	−2
S_N	33.8	0.5	+3
i_N	41.8	?	−10

Mount Wilson

No time marks on N. S.

iP^*_E	11h.49m.50.7s.	Blur only	
iP_{nE}	53.0	0.5	+0.1
$i\overline{P}_E$	56	1.1	+0.2
iS_{nE}	50 38	0.5	+0.6
i_E	39.8	1	+1.4
$i\overline{S}_E$	41.2	1.1	−2.5

READINGS OF SEISMOGRAMS (*Continued*)

GROUP	TIME	(PERIOD) T	A (TRACE AMPLITUDE IN MM.)
Pasadena			
iP^*_N	11h.49m.49.8s.	1.5	+0.1
		0.6	+0.1
iP^*_E	50.5	Blur	
iP_{nN}	53.3	1	+0.2
iP_{nE}	53.9	1.5	−0.2
$i\overline{P}_E$	56.2	2.1	+0.4
$i\overline{P}_N$	56.4	3.1	−0.4
i_N	57.7	0.6	+0.5
i_E	50 36.6	1.5	+0.8
$i\overline{S}_E$	41.4	1.8	+4.5
i_N	44.3	0.8	−2.5
La Jolla			
i_N	11h.49m.21.4s.	1.5	+1
i_E	22	1.7	+1
$i\overline{P}_E$	25.7	1.5?	−0.4
$i\overline{P}_{SE}$	30.7	1	−0.5
$i\overline{P}_{SN}$	30.7	1.2	−0.2
i_E	23.2	1.2	+0.6
i_N	23.2	1.3	−0.5
i_N	24.4	0.5	+1
Tucson			
$i\overline{P}_{SE}$	11h.51m.43s.	Blur	
i_N	53 18	1	−0.1
i_E	21	1	+0.1
$i\overline{S}_N$	29	1	+0.5
i_N	40	8	−1.3

[1] Jeffreys, Harold, "On Two British Earthquakes," *M. N. R. A. S., Geophysical Supplement*, July, 1927.

[2] Nakano, H., "On Rayleigh Wave," Contributions from the *Central Meteorological Observatory of Japan*, **1**, No. 1, 1925.

[3] *Handbuch der Physik* (Geiger u Scheel). **6**, S621.

[4] *Gerland's Beitrage zur Geophysik*, **26**, 1930, pp. 27–33.

[5] *Ibid.*, p. 156.

THE INDEPENDENCE OF DOMINANT SPOTTING AND RECESSIVE SPOTTING ("PIEBALD") IN THE HOUSE MOUSE

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In 1919 Castle[1,2] showed the dominant English spotting pattern of the rabbit to be closely linked with the recessive Dutch spotting pattern, and in 1924 he published evidence proving that both English and Dutch are linked with Angora. In the latter year I began experiments designed to learn whether or not there exists a corresponding linkage between dominant and recessive spotting in the house mouse. Dunn[3] (1920) had previously published data upon the subject. Both Dunn's data and my own collected during three years seemed inconclusive on account of difficulties in classification. It is impossible by inspection to classify correctly all animals containing certain combinations of the dominant and recessive spotting factors and their allelomorphs because most genetic classes overlap in their somatic appearance.

Dr. Geo. D. Snell[4] has shown that recessive piebald (*ss*) and recessive hairless (*hr hr*) are linked with about 10% crossing-over. Thus, recessive hairless (*hr hr*) may be used as a check in the identification of recessive piebald (*ss*) animals if it is introduced into the cross in association with piebald. Piebald animals (*ss*) will then lose their hair at about two weeks after birth whereas animals not containing two doses of piebald will retain their hair, unless a cross-over has occurred. By this control we may be certain of the classification of 90% of the piebald young.

Data employing this check should provide conclusive evidence upon the question of the genetic relationship between dominant and recessive spotting.

The three pairs of gene allelomorphs dealt with in this paper are

W = dominant spotting factor
w = normal allelomorph to *W*
s = recessive "piebald" factor
S = normal allelomorph to *s*
hr = recessive hairless factor
Hr = normal allelomorph to *hr*.

In 1929 I made a cross between a variegated female (*Ww SHr SHr*) and a piebald hairless male (*ww shr shr*). The F_1 offspring were either variegated (*Ww SHr shr*) or self (*ww SHr shr*). The variegated F_1 animals mated *inter se* should produce the following classes, in which cross-over gametes and zygotes are indicated by a star.

	WSHr	**Wshr*	**wSHr*	*wshr*
WSHr	*WSHr* 1 *WSHr*	*Wshr* 2 *WSHr**	*wSHr* 3 *WSHr**	*wshr* 4 *WSHr*
**Wshr*	*WSHr* 5 *Wshr**	*Wshr* 6 *Wshr***	*wSHr* 7 *Wshr***	*wshr* 8 *Wshr**
**wSHr*	*WSHr* 9 *wSHr**	*Wshr* 10 *wSHr***	*wSHr* 11 *wSHr***	*wshr* 12 *wSHr**
wshr	*WSHr* 13 *wshr*	*Wshr* 14 *wshr**	*wSHr* 15 *wshr**	*wshr* 16 *wshr*

Classes 1, 2*, 5*, 6* each contain two doses of *W* and as this combination is lethal they should not appear. Classes 7**, 10** and 11** are due to combinations of two cross-over gametes. The animals obtained may be classified as follows.

GENETIC CLASSES	APPEARANCE	EXPECTED WITH FREE SEGREGATION	FOUND
8*, 14*	Hairless black-eyed white	3.4	1
16	Hairless piebald	1.7	3
4, 13, 3*, 7**, 9*, 10**	Small head or belly spots and variegated	10.2	13
11**, 12*, 15*	Self	5.1	10

There are fewer hairless black-eyed whites and more hairless piebalds than expected, but the discrepancies are relatively too small to be significant and the two classes may overlap. Variegated animals might include two non-crossover classes and four cross-over classes and hence if linkage were present the variegated animals should be less numerous than the expectancy for free segregation shown in the table, but in fact they are more numerous.

The self-progeny consist entirely of animals from cross-over classes and should number about five upon the basis of free segregation. If linkage occurs they should be less than five. Actually they number ten and so give no indication of linkage. Accordingly this result indicates that independent segregation exists between the dominant spotting factor and that recessive "piebald" factor (*s*) which is linked with recessive hairless.

A test of the genetic relationship between *W* and *Shr* is being made, in which no confusion of classes can take place because all spotted animals must contain *W*.

[1] Castle, W. E., "Studies of Heredity in Rabbits, Rats and Mice," *Carn. Inst. Wash. Pub.*, **288**, 1919.

[2] Castle, W. E., "Linkage of Dutch, English and Angora in Rabbits," *Proc. Nat. Acad. Sci.*, **10**, pp. 107–108, 1924.

[3] Dunn, L. C., "Independent Genes in Mice," *Genetics*, **5**, pp. 344–361, 1920.

[4] Snell, G. D., *Ibid.*, **16**, pp. 42–74, 1931,

THE INTERPRETATION OF THE THERMAL DECOMPOSITION OF NITROUS OXIDE

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Unimolecular reactions have heretofore been decompositions or rearrangements of large molecules and it has in general been necessary to assign a considerable number of degrees of freedom to give a theoretical interpretation of the rate. It is, therefore, of considerable interest that the relatively simple molecule, nitrous oxide, has now been shown quite definitely to decompose unimolecularly.[1,2]

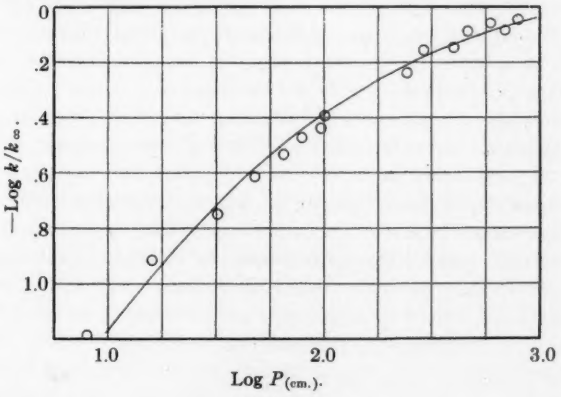

FIGURE 1

As usual the rate is first order at high pressures, but the rate constant falls off at lower pressures. The investigators have shown that the rate as a function of the pressure follows an equation which is the same as that required by Theory I of Rice and Ramsperger. It remains to determine the number of degrees of freedom (squared terms) and molecular diameter required quantitatively to fit the data; a matter of some importance because the unusually simple character of the molecule would not permit the use of more than two or three oscillators and some questionable rotational energy.

We have applied Theories I and II of Rice and Ramsperger[3] and Theory III of Kassel[4] and find that but two squared terms (one classical oscillator) are required to fit the data at 665°C. For this case Theory III is identical with Theory I and Theory II is also practically identical. Figure 1 shows the usual plot of log. k/k_∞ against log P based on Theories II and III.

The curve is theoretical, while the points are experimental. The molecular diameter chosen was 3.05×10^{-8} cm., the critical energy $E = 53{,}000$ cal. p. mol. and $k_\infty = 1.78 \times 10^{-3}$. The molecular diameter chosen is close to the kinetic theory value (3.32×10^{-8} cm.). The interpretation on the basis of Theory III requires that 53,000 calories be the critical energy for the breaking of the bond between a nitrogen atom and an oxygen atom. By using the latest value of the heat of dissociation of the oxygen molecule (118,000 calories) and the heat of formation of the nitrous oxide molecule (17,000 calories) we find a minimum value for the breaking of this bond, namely, $118{,}000/2 - 17{,}000 = 42{,}000$ calories. It is, of course, required only that the experimental value be as great or greater than this figure.

The experimental basis for the interpretation of unimolecular reactions is now better than for bimolecular or trimolecular reactions, a situation quite the contrary to that existing six years ago. Sufficient data for a test of theories is available in at least four cases including such very different molecules as nitrous oxide, which requires but two squared terms and methyl isopropyl di-imide with thirty-three squared terms. The following table gives a list of recognized unimolecular reactions, giving the pressure and temperature ranges that have been studied, the constants in the equation for the rate constant $k_\infty = Ae^{-Q/RT}$ and the number of classical squared terms required by Theory II or III to fit the data. In the case of nitrogen pentoxide Kassel's quantum treatment is necessary to give even a partially satisfactory agreement. In this case a vibrational specific heat of at least 20 cal. p. mol. and a diameter of 17×10^{-8} cm. is required.[5]

DATA FOR UNIMOLECULAR REACTIONS

SUBSTANCE REACTING	P, CM.	T, °C	A	Q CAL. P. MOL	N
C_2H_5CHO[6]	2.0 to 40.0	450 to 600	$>1.38 \times 10^{12}$	54,000	11
$C_2H_5—O—C_2H_5$[7]	2.5 to 50.0	426 to 588	$>3.09 \times 10^{11}$	53,000	> 6
$CH_3—O—CH_3$[8]	3.0 to 90.0	422 to 552	1.52×10^{13}	58,500	11
$CH_3—N{=}N—CH_3$[9]	0.026 to 70.79	278 to 327	1.07×10^{16}	51,200	25
$CH_3—N{=}N—C_3H_7$[10]	0.0058 to 13.12	250 to 332	2.80×10^{15}	47,480	33
$C_3H_7—N{=}N—C_3H_7$[11]	0.025 to 4.60	250 to 290	5.6×10^{13}	40,900	>40
$CH_3—N{=}N—NHCH_3$[12]	0.019 to 8.0	200 to 230	4.05×10^{11}	33,800	14
d-pinene[13]	17.0 to 116.0	184 to 237	5.40×10^{14}	43,700	>20
Ethylene oxide[14]	2.4 to 90.5	378 to 445	9.9 to 10^{12}	52,000	>14
N_2O_5[15,16,17]	0.00042 to 70.0	0 to 65	4.5×10^{13}	24,700	30[5]
N_2O[1,2]	8.1 to 800	560 to 667	4.0×10^{9}	53,000	2

[1] Volmer and Kummerow, *Zeit. Phys. Chem.*, **B9**, 141 (1930).
[2] Nagasako and Volmer, *Ibid.*, **B10**, 414 (1930).
[3] Rice and Ramsperger, *J. Am. Chem. Soc.*, **49**, 1617 (1927).
[4] Kassel, *J. Phys. Chem.*, **32**, 225 (1928).
[5] Kassel, *J. Am. Chem. Soc.*, **52**, 3972 (1930).
[6] Hinshelwood and Thompson, *Proc. Roy. Soc.*, **113A**, 221 (1926).
[7] Hinshelwood, *Ibid.*, **114A**, 84 (1927).

[8] Hinshelwood and Askey, *Proc. Roy. Soc.*, **115A**, 215 (1927).
[9] Ramsperger, *J. Am. Chem. Soc.*, **49**, 912, 1495 (1927).
[10] Ramsperger, *Ibid.*, **51**, 2134 (1929).
[11] Ramsperger, *Ibid.*, **50**, 714 (1928).
[12] Ramsperger and Leermakers, forthcoming.
[13] D. F. Smith, *J. Am. Chem. Soc.*, **49**, 43 (1927).
[14] Heckert and Mack, *Ibid.*, **51**, 2706 (1929).
[15] Daniels and Johnson, *Ibid.*, **43**, 53 (1921).
[16] Ramsperger and Tolman, *Proc. Nat. Acad. Sci.*, **16**, 6 (1930).
[17] Schumacher and Sprenger, *Ibid.*, **16**, 129 (1930).

REVERSIBLE COAGULATION IN LIVING TISSUE. I[1]

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In a previous paper Bancroft and Richter[3] have confirmed the hypothesis of Claude Bernard that anesthesia is due to the reversible coagulation of the colloids of the sensory nerves. It appeared that the proteins were the important factors rather than the lipoid materials. If we have different proteins in different parts of the body we shall expect them to be flocculated reversibly to different extents by different drugs. Claude Bernard pointed out that the anesthetics affect the brain first but the sensory nerves very soon after, while the narcotics affect the brain much more readily than the sensory nerves. Morphine will, therefore, tend to drive chloroform from the brain to the sensory nerves, thereby increasing the anesthetic effect of the chloroform.

Histamine apparently acts primarily on the sympathetic nerves of the body, while curare acts primarily on the nerves connected with the voluntary muscles, paralyzing the victim without necessarily causing him to lose consciousness. From our point of view, it is immaterial for the moment whether curare and the quaternary organic nitrogen salts coagulate the proteins of the nerve or the proteins of a substance between the nerve and the muscle, as Bayliss[4] suggests.

"The two excitable substances, already discussed, muscle and nerve, are not the only members of the class. It might be supposed that a nerve acting on a muscle merely formed some kind of direct connection therewith; but a simple experiment shows that there is something between them, itself an excitable substance. Let us take two nerve-muscle preparations and arrange the nerves of both on similar electrodes in series in the same circuit, so that they can be excited with the same strength of stimulus. On the one nerve, between the seat of excitation and the muscle, we place a

pair of non-polarizable electrodes, through which we send a galvanic current of sufficient strength to block the nerve impulses on the side. Both nerves are thus excited, one muscle only. After a time, the muscle becomes fatigued and ceases to respond. At this moment, the galvanic current causing the block is cut off; the muscle on this side goes into tetanus. In this way, it is seen that the fatigue is not localized in the nerve trunk. The next step is to apply electrodes directly to the muscle which had ceased contracting; it is found to be able to respond vigorously. So that the *seat of the fatigue* is not in the actual contractile substance of the muscle. The unavoidable conclusion is that there is some intermediate substance, more easily fatigued than either nerve or muscle.

"The action of the arrow poison, *curare*, affords similar evidence. If the nerve only is immersed in a solution of this drug, it is not paralyzed. If the muscle is immersed, excitation of the nerve has no effect upon it; but it is not because the muscle itself is paralyzed, since placing the electrodes on it produces contraction."

Since Bayliss wrote these paragraphs, Lapicque has shown that curare changes the chronaxie of the muscle, thereby throwing it out of step with the nerves. It is a blocking effect and not a complete paralysis of the muscle.

From the modern viewpoint of the large number of proteins and the much larger number of mixtures of proteins, it is evident that Claude Bernard's theory of narcosis leads directly to the existence of all sorts of ailments due to the selective coagulation of different proteins by different drugs. It will be a long time before we know what the proteins are in different parts of the body and what the coagulating action of different drugs is; but it is desirable to recognize this as a major problem in research. For instance, what are the proteins or what is the predominant protein in the sympathetic nerves, flocculated by histamine? At present, nobody knows and we do not know what protein in the brain is flocculated more readily by anesthetics than the predominating protein in the sensory nerves.

It is probable that this problem can best be attacked indirectly, by studying the behavior of different known proteins with different drugs and then deducing what is the predominant protein in any part of the nervous system from the action of the drug which flocculates any given protein most readily. This is only another way of wording what was said nearly seventy-five years ago by Claude Bernard.[5] "I have looked upon toxic substances as physiological instruments more delicate than our mechanical methods and destined to dissect, so to speak, the anatomical elements of the living organism. I have considered them as real life-reagents."

One way in which we can tell that we are dealing with a reversible coagulation of nerve proteins is that a substance which peptizes proteins must counteract to some extent the effect of any drug which produces

primarily a flocculation of proteins in living tissue, quite regardless of what that effect may be, always assuming that the substance used does not give rise to other disturbing effects.

As an instance of the latter we may take potassium iodide which is used to some extent medicinally. Since it peptizes gelatin and albumin very strongly, one would expect it to counteract morphine or chloroform to some extent. On the other hand the iodide may have distinctly harmful effects as has been pointed out by Dr. Keith.[6]

"A number of observers in Europe and on this continent have drawn attention to the harmful effects that may result under some circumstances from the use of iodine. They have found that when it is employed for the prevention of goiter without due medicinal supervision, used in quack medicines or taken as iodized salt in irregular amounts for a long period, serious toxic symptoms may be provoked.

"Birker of Aarau, Switzerland, states that the carrying out of systematic iodine treatment has its dangers. Although he favors the use of iodine as a preventive measure, he thinks that it will take almost a lifetime before we know its action completely. He states that the immediate result in Switzerland has been a great increase in toxic goiters; that at least 2000 Swiss people have consulted physicians for the symptoms provoked by iodized salt; and that, in his own clinic, where operations for goiter formerly showed 2 to 5 per cent of cases with toxic symptoms, now his records show that 20 to 25 per cent are toxic cases; some clinics in that country report as high as 50 per cent."

Rabuteau[7] claims that all potassium salts are muscle poisons. When injected into the veins of a dog, death occurred with low concentrations of potassium salts and no effect with much higher concentrations of sodium salts. Some of Rabuteau's data are given in Table 1.

TABLE 1

INJECTIONS IN THE VEINS OF A DOG

SALT	DEATH GRAMS	SALT	NO EFFECT GRAMS
K_2SO_4	1	Na_2SO_4	14
K_2HPO_4	1.5	Na_2HPO_4	8
KNO_3	1	$NaNO_3$	5
KNO_2	1	$NaNO_2$	2.5
KI	2	NaI	5
K_2CO_3	1.5	Na_2CO_3	7

It seemed wise, therefore, to make preliminary experiments with sodium thiocyanate, NaCNS, because the sodium ion is not especially toxic and the thiocyanate ion is very effective in causing the peptization of proteins. The difference in action between sodium and potassium thiocyanate is apparently very great. We have injected 30 cc. of a 20% sodium thio-

cyanate solution (six grams) intravenously into a dog without any mishap. On the other hand Claude Bernard[8] has made comparative experiments on curare, strychnine and potassium thiocyanate.

"Of the three toxic substances of which we have compared the actions, the most fatal is without question potassium thiocyanate. We have seen it strike down vertebrate animals through a sudden stoppage of the heart and later of all the muscles; the effects are definite and irremediable. The muscular fibre which belongs to the organic or to the animal system is destroyed in all its functions with invertebrates just as with vertebrates. The muscular fibre is then an anatomical element enjoying everywhere identical properties, capable of characterizing it physiologically."

Experiments were made mostly with rabbits weighing 600–2500 grams. The usual treatment was the intravenous injection of 0.8 cc. or more of a ten per cent solution of sodium thiocyanate. All intravenous injections were made rapidly either in the marginal or central vein of the ear. A rabbit weighing 600 grams was anesthetized with ether completely enough so that the lid reflex was lost. The process was stopped as soon as the reflex was lost—3 minutes and 20 seconds by a stop-watch. In $4^1/_2$ minutes after the cone was removed the lid reflex returned. In $6^1/_2$ minutes the rabbit stood up of its own accord. After several hours the process was repeated. This time the ether was administered for 4 minutes 10 seconds. During the last minute that the ether was being administered, 0.9 cc. of a ten per cent solution of sodium thiocyanate was injected intravenously. The lid reflex returned 50 seconds after the mask was removed. The rabbit arose of its own accord $4^1/_2$ minutes after the mask was removed. It was, however, very intoxicated. The difference between 50 seconds and $4^1/_2$ minutes for the return of the lid reflex is as much as one could expect; but people might claim that it was the disturbance due to the injection that produced the result and not the peptizing action of the sodium thiocyanate. This could have been checked by injecting the same amount of some other solution; but we preferred to test the matter in another way by using an anesthetic with a longer time of recovery, amytal.

Into a rabbit weighing about 600 grams there was injected intravenously 0.03 gram of amytal dissolved in a sodium carbonate solution. After four minutes 1.0 cc. of a ten per cent solution of sodium thiocyanate was injected intravenously. The rabbit raised its head and worked its legs during the injection. Fifteen minutes later the lid reflex returned. After twenty-five minutes 0.5 cc. of the sodium thiocyanate was injected subcutaneously. So far as we were able to tell, this brought about consciousness. Several times thereafter 0.5 cc. portions of sodium thiocyanate solution were injected subcutaneously. The animal never lost consciousness after the first injection of sodium thiocyanate under the skin, and from that

time on it gradually gained in strength. It was four hours and fourteen minutes before the control rabbit regained consciousness after receiving an intravenous injection of 0.03 gram amytal followed by no injection of sodium thiocyanate.

In all the experiments with morphine, the hydrochloride was used. A water solution of morphine hydrochloride containing 0.12 gram morphine was injected subcutaneously into a rabbit weighing 2300 grams. The animal became dull after four minutes and was asleep in twelve minutes. There were some signs of recovery in forty-one minutes; in fifty-nine minutes the rabbit tried to rise. After one hour and fourteen minutes the rabbit was not able to sit up. Almost immediately thereafter there was a lapse back into unconsciousness. The respiration was twelve per minute after three hours and eight minutes. Shaking produced no visible effect upon the narcosis. After three hours and fifty-six minutes the rabbit was turned over on its back. It stayed that way with its head thrown back and its legs at four different angles for ten minutes, after which it was turned over on its side. There was no defecation up to this time. The rabbit showed practically no resistance to severe mauling after four hours and twenty-six minutes; it kicked once upon being lifted by the ears. The respiration was only eighteen per minute after five hours. The next morning the rabbit was normal though somewhat weak. This was the control experiment.

Another rabbit, of practically the same age and weighing 2500 grams, received a subcutaneous injection of 0.123 gram morphine in water solution. The animal became dull in six minutes and was asleep in ten minutes. When the narcosis became deep and definite, and the breathing was very slow, 3.8 cc. of a ten per cent solution of sodium thiocyanate was injected intravenously. Eight minutes later the animal became conscious. It was able to sit up without any trouble, but was in a dull stupor. After one hour and three minutes from the time that the morphine was injected, the animal lapsed back somewhat; so 3 cc. of a ten per cent solution of sodium thiocyanate was injected intravenously. This brought the animal back quickly to consciousness. After two hours and thirty-five minutes the respiration was twenty-two per minute as compared to twelve per minute for the control rabbit. After three hours and twenty-seven minutes the rabbit was turned over on its back; it kicked hard and turned over on its side. The experiment was repeated with the same result. It was repeated again, accompanied by an attempt to soothe the rabbit by stroking its throat and chest, which generally puts a normal rabbit to sleep. Despite this the rabbit kicked hard and determinedly until it was freed. Up to this time the rabbit had eliminated feces quite a few times, whereas the control rabbit had not done so at all. This difference became more marked as time went on. After four hours this rabbit put up a strong resistance to mauling

as compared to no resistance by the other one. It kicked violently upon being lifted by the ears. In four and one-half hours the respiration was thirty-eight per minute, whereas the respiration of the control rabbit was only eighteen per minute after five hours. So far as we could determine, the second rabbit never lost consciousness after being brought out of unconsciousness by the first injection of sodium thiocyanate.

Into a control rabbit weighing about one kilogram, there was injected 1.0 cc. of a strychnine solution containing 0.03 gram strychnine per 100 cc. The animal died in eight minutes. Into a second rabbit of practically the same weight we injected intravenously 1.1 cc. of a ten per cent sodium thiocyanate solution. Seven minutes later 1.0 cc. of the strychnine solution was injected, also intravenously. The rabbit was affected by severe spasmodic tetanic convulsions which were not so severe as in the control rabbit, which showed very severe dyspnea as against none for the test rabbit. The tetanic stimulation finally knocked the test rabbit down. This was followed by disappearance of the tetanic convulsions and the subsequent appearance of severe clonic movements which brought the rabbit into an upright position. From then on the rabbit seemed merely to be depressed. The time elapsed from injection of the strychnine to practically complete recovery was twenty minutes, as against death in eight minutes for the control rabbit.

Experiments were also made on anaphylaxis. Since young rabbits are said to be more sensitive to anaphylactic shock, the experiments were made with young rabbits weighing about 800 grams. A rabbit was sensitized by a subcutaneous injection of 1.0 cc. of a twenty-two per cent egg-white sol. Twenty-one days later, 0.8 cc. of a ten per cent sodium thiocyanate solution was injected intravenously. This was followed twenty minutes later by the intravenous injection of 2.0 cc. of a twenty per cent egg-white sol. Absolutely no symptoms of shock were noted, while the control rabbit suffered a severe anaphylactic shock.

Since histamine kills a non-sensitized rabbit, 1.1 cc. of a ten per cent sodium thiocyanate solution was injected intravenously into a rabbit weighing 790 grams. Fourteen minutes later, 0.85 cc. of a histamine solution containing 3.0 milligrams histamine per cubic centimeter was injected intravenously. There were no noticeable ill effects at all, whereas a much smaller dose of histamine killed in seven minutes a slightly larger rabbit, which had not been protected by sodium thiocyanate.

The general conclusions of this paper are:

1. In accordance with the views of Claude Bernard, the effect of ether, amytal, morphine, and histamine is to coagulate certain nerve colloids, apparently the proteins.

2. Anaphylactic shock is due to the coagulation of certain nerve proteins, chiefly those of the sympathetic nerves.

3. Since sodium thiocyanate peptizes proteins, it should alleviate and perhaps counteract all disturbances due to coagulation of the nerve proteins.

4. It has been shown experimentally that intravenous injections of sodium thiocyanate solutions bring rabbits out of the unconsciousness due to ether, amytal or morphine more rapidly than is normal. The rabbits breathe approximately twice as rapidly under these conditions.

5. It has been shown experimentally that intravenous injections of sodium thiocyanate solutions into rabbits can prevent death from strychnine or histamine; and can prevent anaphylactic shock in rabbits previously sensitized by subcutaneous injection of an egg-white sol.

6. Potassium thiocyanate cannot be substituted safely for sodium thiocyanate because of the greater toxic action of potassium salts. This has been known for over half a century and yet many medical men give potassium and sodium salts interchangeably. The only justification for this is the as yet unproven assumption that potassium salts do not affect human beings as they do dogs and rabbits.

[1] This work is part of the programme now being carried out at Cornell University under a grant from the Heckscher Foundation for the Advancement of Research established by August Heckscher at Cornell University.

[2] Eli Lilly Research Fellow.

[3] Bancroft and Richter, *Proc. Nat. Acad. Sci.*, **16**, 573 (1930).

[4] Bayliss, *Principles of General Physiology*, 399 (1915).

[5] Claude Bernard, *Leçons sur les effets des substances toxiques et médicamenteuses*, 5, (1857.)

[6] Keith, *Canadian Med. Assn. J.*, **16**, 1171 (1926).

[7] Rabuteau, *Éléments de toxicologie*, 541 (1887).

ON THE UNIFIED FIELD THEORY. IV

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In this note[1] we shall consider the characteristic lines or bicharacteristics (Hadamard) determined by the differential equation of the characteristic surfaces. The bicharacteristics are the so-called geodesics of zero length which give the light tracks in the Einstein theory of gravitation.[2] It is shown that a characteristic surface is generated by the totality of bicharacteristics issuing from a point P of the continuum. In a system of metric local coördinates (see Sect. 3) this characteristic surface has the form

$$\sum_{k=1}^{4} e_k w^k w^k = 0$$

and so appears to an observer in the local system as a spherical wave propagated with unit velocity.

While local coördinates were first mentioned in connection with the Unified Theory in Note I, my attention has been called since writing that note to the fact that such coördinates are not new in mathematical literature. In 1901 Poincaré published a paper ("Quelques remarques sur les group continus," *Rend. Circ. Matem. Palermo*, **15**, p. 321) in which he introduced a set of quantities of the character of affine local coördinates (see Sect. 3). More recently a similar system of variables was used by J. A. Schouten ("Kontinuierliche Transformationsgruppen," *Math. Ann.*, **102** (1929), p. 244). Also A. D. Michal ("Scalar Extensions of an Orthogonal Ennuple of Vectors," *Am. Math. Monthly*, **37** (1930), p. 529) has defined a system of geodesic coördinates S_k which become identical with the metric local coördinates when k is allowed to become indefinitely large.

1. In Note III a three-dimensional surface

$$\Phi \equiv x^1 - \varphi(x^2, x^3, x^4) \tag{1.1}$$

was called characteristic provided that the equation

$$g^{\alpha\beta} \frac{\partial \Phi}{\partial x^\alpha} \frac{\partial \Phi}{\partial x^\beta} = 0 \tag{1.2}$$

was satisfied over this surface. If we substitute (1.1) into (1.2) this equation becomes

$$\boxed{F \equiv g^{11} - 2 \sum_{\beta=2}^{4} g^{\beta 1} p_\beta + \sum_{\alpha=2}^{4} \sum_{\beta=2}^{4} g^{\alpha\beta} p_\alpha p_\beta = 0} \tag{1.3}$$

where

$$p_\alpha = \frac{\partial \varphi}{\partial x^\alpha} \qquad (\alpha = 2, 3, 4).$$

The characteristic surfaces (1.1) are defined by the differential equation (1.3).

If we put

$$P^\alpha = \frac{\partial F}{\partial p_\alpha} \quad (\alpha = 2, 3, 4)$$

then

$$\frac{dx^2}{P^2} = \frac{dx^3}{P^3} = \frac{dx^4}{P^4} = \frac{dx^1}{\sum_{\alpha=2}^{4} p_\alpha P^\alpha}, \tag{1.4}$$

defines a system of curves on the characteristic surface (1.1). Through any point P of the surface (1.1) there passes in general one and only one

curve C of the system (1.4); these curves C are the bicharacteristics determined by the field equations of this theory.[3]

2. We have supposed the equation of the characteristic surfaces to be solved for the coördinate x^1 in the previous work. Let us now take this equation in the more general form

$$\pi(x^1, \ldots, x^4) = 0 \tag{2.1}$$

subject to the condition that the function π depends explicitly on the x^1 coördinate. Then from (2.1) we have

$$p_\alpha = -\frac{q_\alpha}{q_1} \qquad (\alpha = 2, 3, 4) \tag{2.2}$$

where $$q_\alpha = \frac{\partial \pi}{\partial x^\alpha} \qquad (\alpha = 1, 2, 3, 4).$$

Hence

$$q_1^2 F = g^{\alpha\beta} q_\alpha q_\beta. \tag{2.3}$$

Also $$-q_1 P^\alpha = 2g^{\alpha\beta} q_\beta \qquad (\alpha = 2, 3, 4);$$

and $$-q_1 \left(\sum_{\alpha=2}^{4} p^\alpha P^\alpha \right) = 2g^{1\beta} q_\beta$$

when the condition

$$g^{\alpha\beta} q_\alpha q_\beta = 0 \tag{2.4}$$

is taken into account. The set of equations (1.4) can therefore be given the more symmetrical form

$$\frac{dx^\alpha}{dv} = 2g^{\alpha\beta} q_\beta. \tag{2.5}$$

On solving equations (2.5) for the quantities q_α we obtain

$$q_\alpha = {}^1/_2 \, g_{\alpha\beta} \frac{dx^\beta}{dv}. \tag{2.6}$$

Now put

$$dq_\alpha = q_{\alpha\beta} \, dx^\beta \tag{2.7}$$

so that $q_{\alpha\beta}$ is the second derivative of the function $\pi(x)$, and then differentiate (2.4) with respect to x^γ where $\gamma = 2, 3, 4$. This gives

$$\frac{\partial g^{\alpha\beta}}{\partial x_\gamma} q_\alpha q_\beta + 2g^{\alpha\beta} q_{\alpha\gamma} q_\beta = \left(\frac{\partial g^{\alpha\beta}}{\partial x^1} q_\alpha q_\beta + 2g^{\alpha\beta} q_{\alpha 1} q_\beta \right) \frac{q_\gamma}{q_1}.$$

$$(\gamma = 2, 3, 4)$$

Making use of (2.5) and (2.7) this last system of equations becomes

$$\frac{dq_\gamma}{dv} + \frac{\partial g^{\alpha\beta}}{\partial x^\gamma} q_\alpha q_\beta = \left(\frac{dq_1}{dv} + \frac{\partial g^{\alpha\beta}}{\partial x^1} q_\alpha q_\beta\right) \frac{q^\gamma}{q_1}. \tag{2.8}$$

$$(\gamma = 2, 3, 4)$$

Now differentiate the equations (2.5), which are satisfied along a bicharacteristic C on the characteristic surface (2.1), with respect to the parameter v; from the equations so obtained eliminate the q_α and the derivatives of the q_γ by means of (2.6) and (2.8), respectively. We thus obtain a set of equations which takes the form

$$\frac{d^2x^\alpha}{dv^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{dv} \frac{dx^\gamma}{dv} = \frac{1}{q_1}\left(\frac{\partial q_1}{\partial x^\beta} \frac{dx^\beta}{dv} - \frac{1}{4} \frac{\partial g_{\beta\gamma}}{\partial x^1} \frac{dx^\beta}{dv} \frac{dx^\gamma}{dv}\right) \frac{dx^\alpha}{dv}, \tag{2.9}$$

where the $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols derived from the components $g_{\alpha\beta}$. At a point P with coördinates x^α_0 on a bicharacteristic C, equations (2.5) determine the values of the derivatives of the x^α with respect to the parameter v, i.e.,

$$\left(\frac{dx^\alpha}{dv}\right)_0. \tag{2.10}$$

If for $v = 0$, we suppose that $x^\alpha = x^\alpha_0$ and the derivatives of the x^α with respect to the parameter v have the above values (2.10) then equations (2.9) determine a solution

$$x^\alpha = E^\alpha(v) \tag{2.11}$$

representing the bicharacteristic C. By a change of the parameter v equations (2.9) can be given a simpler form. In fact, under a transformation of parameter (2.9) becomes

$$\frac{d^2x^\alpha}{du^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{du} \frac{dx^\gamma}{du} = -\frac{\left[\frac{d^2u}{dv^2} + \frac{1}{q_1}\left(\frac{1}{4} \frac{\partial g_{\beta\gamma}}{\partial x^1} \frac{dx^\beta}{dv} \frac{dx^\gamma}{dv} - \frac{\partial q_1}{\partial x^\beta} \frac{dx^\beta}{dv}\right) \frac{du}{dv}\right]}{\left(\frac{du}{dv}\right)^2} \frac{dx^\alpha}{du}. \tag{2.12}$$

Choosing the new parameter u to be such a function of v that the bracket expression in the right member of (2.12) vanishes along C, we have

$$\boxed{\frac{d^2x^\alpha}{du^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{du} \frac{dx^\gamma}{du} = 0} \tag{2.13}$$

along the bicharacteristic C. Since the condition

$$\boxed{g_{\alpha\beta}\frac{dx^\alpha}{du}\frac{dx^\beta}{du}=0} \tag{2.14}$$

is satisfied on account of (2.4) and (2.6) along C, this curve is of the type commonly described as a geodesic of zero length. Moreover, the fact that the equation (2.1) of the characteristic surface depends explicitly on the x^1 coördinate is in no way involved in (2.13) and (2.14) so that these latter equations must be satisfied along the bicharacteristics on any characteristic surface regardless of the particular form $\pi(x) = 0$ of the equation by which the characteristic surface is defined.

3. In Note I we introduced a system of local coördinates z^i which possessed the property that the equations of a path through the origin of the local system had the form of the parametric equations of a Euclidean straight line through the origin of a system of rectangular cartesian coordinates. As the paths are determined by the affine connection of the continuum it seems natural to refer to the above coördinates as *affine local coördinates*. The equations (2.13) to which we are led by the foregoing work suggest that we likewise construct a system of *metric local coördinates* defined by the metric properties of the continuum in an analogous manner. More precisely, a system of metric local coördinates w^i will be characterized geometrically by the postulates in Sect. 2 of Note I except for the fact that postulate D regarding the paths of the continuum will be replaced by a similar postulate involving the geodesics (2.13). By a method similar to that employed in Note I it can be shown (1) *that the metric local coördinates w^i remain unchanged when the underlying coördinates x^α undergo an arbitrary analytic transformation T, i.e.,*

$$\boxed{w^i=\bar{w}^i},$$

and (2) *that when the fundamental vectors h_i^α undergo an orthogonal transformation*

$$*h_k^\alpha = a_k^i\, h_i^\alpha \tag{3.1}$$

the metric local coördinates associated with any point P likewise undergo an orthogonal transformation, i.e., a linear homogeneous transformation,

$$\boxed{w^i = a_k^i\, w_*^k}$$

which leaves the form

$$\sum_{k=1}^{4} e_k w^k w^k \tag{3.2}$$

invariant. The local coördinates w^i are related to the underlying coordinates x^α by a transformation of the form

$$x^\alpha = p^\alpha + h_i^\alpha(p)w^i - \frac{1}{2!}\Gamma_{ij}^\alpha(p)w^i w^j - \frac{1}{3!}\Gamma_{ijk}^\alpha w^i w^j w^k - \ldots$$

in which the coefficients I are determined by successive differentiation of the equations

$$\left(\frac{\partial^2 x^\alpha}{\partial w^j \partial w^k} + \Gamma_{\beta\gamma}^\alpha \frac{\partial x^\beta}{\partial w^j}\frac{\partial x^\gamma}{\partial w^k}\right) w^j w^k = 0$$

and evaluation at the origin of the local system.

If we transform the components of a tensor $T_{\mu\ldots\nu}^{\alpha\ldots\beta}(x)$ to a system of metric local coördinates thereby obtaining $t_{|k\ldots m|}^{|i\ldots j|}(w)$ and evaluate the components t at the origin of this system, we obtain a set of absolute invariants $T_{k\ldots m}^{i\ldots j}$ under transformations of the x^α coördinates. More generally, if we differentiate the components t any number of times and evaluate at the origin of the (w) system, we obtain a set of quantities, namely,

$$T_{k\ldots m|p\ldots q}^{i\ldots j} = \left(\frac{\partial^r t_{|k\ldots m|}^{|i\ldots j|}}{\partial w^p \ldots \partial w^q}\right)_{w=0}, \tag{3.3}$$

each of which is an absolute invariant with respect to transformations of the x^α coördinates. Let us denote the components h_α^i and $g_{\alpha\beta}$ by $B_{|j|}^i$ and $C_{|ij|}$, respectively, in a system of metric local coördinates. Then the equations

$$h_{j|k\ldots m}^i = \left(\frac{\partial^r B_{|j|}^i}{\partial w^k \ldots \partial w^m}\right)_{w=0} \tag{3.4}$$

and

$$g_{ij|k\ldots m} = \left(\frac{\partial^r C_{|ij|}}{\partial w^k \ldots \partial w^m}\right)_{w=0} \tag{3.5}$$

define sets of absolute invariants h and g under arbitrary analytic transformations of the x^α coördinates. When the fundamental vectors undergo an orthogonal transformation (3.1) the absolute invariants (3.3), (3.4) and (3.5) transform by the tensor transformation implied by the indices in the symbol of the invariant.

The explicit formulae for any of the invariants (3.3), (3.4) and (3.5)

as well as the identities satisfied by the invariants (3.4) and (3.5) can easily be deduced.

4. It is a known fact that the totality of characteristics through a point P of the continuum of a partial differential equation of the first order, generates a characteristic surface.[3] Hence an integral surface of (1.3) is generated by the totality of characteristics through P of this equation. When the equation (1.3) is taken with reference to a system of metric local coördinates and the point P is at the origin of this system, the discussion of this integral surface is of especial interest.

Denoting the components $g^{\alpha\beta}$ by $C^{|ij|}$ when taken with respect to a system of metric local coördinates the equation (1.3) becomes

$$C^{|11|} - 2\sum_{i=2}^{4} C^{|1i|}r_i + \sum_{i=2}^{4}\sum_{j=2}^{4} C^{|ij|}r_i r_j = 0, \tag{4.1}$$

where

$$r_i = \frac{\partial w^1}{\partial w^i} \quad (i = 2, 3, 4). \tag{4.2}$$

The geodesics (2.13) through the origin have equations

$$w^i = \eta^i u \tag{4.3}$$

in which the η^i are arbitrary constants; hence the condition that (2.13) should be a bicharacteristic of the field equations, i.e., the condition that (2.14) be satisfied along (2.13), is

$$\sum_{i=1}^{4} e_i \eta^i \eta^i = 0. \tag{4.4}$$

It is seen that $\eta^1 \neq 0$ since if this were not true it would follow from (4.4) that the remaining quantities η^i would likewise vanish. The normalizing condition $\eta^1 = 1$ can therefore be imposed. Equation (4.4) then becomes

$$(\eta^2)^2 + (\eta^3)^2 + (\eta^4)^2 = 1. \tag{4.5}$$

Taking

$$\eta^2 = \sin\theta\cos\varphi,\ \eta^3 = \sin\theta\sin\varphi,\ \eta^4 = \cos\theta \tag{4.6}$$

the condition (4.5) is satisfied automatically. With these values of the η's equations (4.3) for $i = 2, 3, 4$ are the equations of transformation between a system of spherical coördinates θ, φ, u and a system of rectangular cartesian coördinates w^2, w^3, w^4.

We shall have occasion to use the following identities[4]

$$w^1 = C_{|1j|}w^j,\ w^k = -C_{|kj|}w^j \quad (k = 2, 3, 4). \tag{4.7}$$

From these identities we have that

$$1 = C_{|1j|}\eta^j, \eta^k = -C_{|kj|}\eta^j \ (k = 2, 3, 4),$$

if we take the point with coördinates w^i to lie on one of the geodesics of zero length passing through the origin.

A set of functions $w^i (i = 1, 2, 3, 4)$ and $r_i (i = 2, 3, 4)$ depending on three independent variables is said to constitute an integral of (4.1) in the general sense of Sophus Lie (1) if the functions w^i and r_i satisfy (4.1) and (2) if

$$dw^1 = \sum_{i=2}^{4} r_i dw^i. \tag{4.8}$$

We can readily construct an integral of (4.1) in this sense. Let us put $r_i = \eta^i (i = 2, 3, 4)$, thereby defining the r_i as functions of the variables θ, φ on account of (4.6). Equations (4.3) define the w^i as functions of the variables θ, φ, u. At the origin, i.e., when $u = 0$, equations (4.1) are satisfied since these equations reduce to (4.5). Using (4.7) equations (4.1) can be put into the form

$$C_{|ij|}\eta^i\eta^j = 0; \tag{4.9}$$

in fact, the left members of (4.1) and (4.9) are identically equal. Now the above functions w^i and r_i satisfy (4.1). This is seen from the fact (1) that (4.9) is satisfied for $u = 0$ and (2) that the left member of (4.9) is invariant along a geodesic of zero length issuing from the origin. To show that (4.8) is also satisfied we observe that the differential du is identical with the left member of this expression; the right member of (4.8) likewise becomes equal to du when account is taken of (4.3) and (4.5). Hence the above functions w^i and r_i of the independent variables θ, φ, u constitute an integral of (4.1) in the sense of Lie.

From (4.3) for $i = 2, 3, 4$ and (4.5) we have

$$u = \pm \sqrt{(w^2)^2 + (w^3)^2 (w^4)^2}.$$

Hence (4.3) for $i = 1$ gives

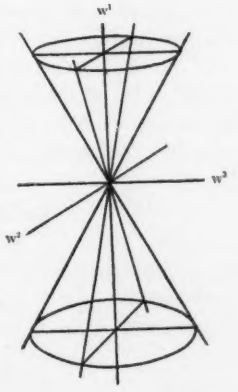
$$\boxed{w^1 = -\sqrt{(w^2)^2 + (w^3)^2 + (w^4)^2}} \tag{4.10}$$

or

$$\boxed{w^1 = +\sqrt{(w^2)^2 + (w^3)^2 + (w^4)^2}}\ . \tag{4.11}$$

The parametric equations (4.3) are equivalent to (4.10) for $u \leqq 0$ and to (4.11) for $u \geqq 0$. By elimination of the variables θ, φ, u from the above integral formed by w^i and r_i it follows immediately that both (4.10) and (4.11) constitute an integral of (4.1) in the ordinary sense where the quantities r_i are defined by (4.2).

The complete integral surface which is the graph of (4.10) and (4.11) is a cone with center at the origin. This is illustrated in the accompanying figure in which the coördinate w^4 is suppressed for the purpose of graphical representation.



If w^1 has a constant negative value (4.10) represents a sphere with center at the origin of a system of rectangular cartesian coördinates w^2, w^3, w^4; similarly (4.11) is the equation of a spherical surface with center at the origin if w^1 is taken as a positive constant. Hence if w^1 increases algebraically starting with some negative value the complete integral surface composed of (4.10) and (4.11) appears as a series of moving concentric spheres which first shrink down to a point, namely, the origin, and then expand. Introducing the coördinate w^1 as the time coördinate, the above surface is seen to shrink or expand with unit velocity.

In a later communication we shall investigate the possibility of interpreting the integral surface (4.10) or (4.11) as a wave surface in the four-dimensional continuum.

[1] Previous notes have appeared in these *Proceedings*, **16**, pp. 761–776 and pp. 830–835, 1930; also, **17**, pp. 48–58, 1931.

[2] T. Levi-Civita, "Caratteristiche e bicharatteristiche dellee quazioni gravitazionali di Einstein," *Rendiconti Accad. Lincei*, **11**, pp. 1–11, 1930; *Ibid.*, pp. 113–121.

[3] E. Goursat, *Leçons sur l'integration der équations aux dérivées partielles du premier ordre*, 2nd ed., 1921, pp. 184–192.

[4] These identities can be derived in a manner similar to the analogous identities in normal coördinates (Riemann-Birkhoff). See O. Veblen, "Invariants of Quadratic Differential Forms," No. 24, Cambridge (1927), p. 96.

THE ARITHMETIC OF POLYNOMIALS IN A GALOIS FIELD

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1. Let p be any prime (including 2) and π any positive integer; let $GF(p^\pi)$ be the Galois field of order p^π. We define[1] $\mathfrak{D}(p^\pi, x)$ as the totality of polynomials in an indeterminate, x, with coefficients in the $GF(p^\pi)$. In this paper we consider some of the arithmetic properties of the polynomials in $\mathfrak{D}$.

2. If M is a polynomial in $\mathfrak{D}$, $sgn\ M$ is the coefficient of the highest power of x; if $sgn M = 1$, M is primary. If M is of degree μ, $|M|$ by definition $= p^{\pi\mu}$. Now the number of primary polynomials of degree μ is $p^{\pi\mu}$. Accordingly we define the ζ-function in $\mathfrak{D}$ by

$$\zeta(s) = \sum_F \frac{1}{|F|^s} = \prod_P \left(1 - \frac{1}{|P|^s}\right)^{-1} \qquad (s > 1),$$

F running over all primary polynomials, P over all primary irreducible polynomials. It is easily verified that

$$\zeta(s) = \frac{1}{1 - p^{\pi(1-s)}}.$$

3. We now define a number of numerical functions of a primary argument F (except (*vi*) which is of a different nature)

$$\mu(1) = 1,\ \mu(F) = 0 \text{ for } P^2 \mid F, \tag{i}$$

$$\mu(F) = (-1)^\rho, \text{ for } F = P_1 \ldots P_\rho,$$

P_i irreducible and distinct.

$$\lambda(1) = 1, \tag{ii}$$

$$\lambda(F) = (-1)^{\alpha_1 + \cdots + \alpha_\rho} \text{ for } F = P_1^{\alpha_1} \ldots P_\rho^{\alpha_\rho}$$

$\tau(F)$ is the number of primary divisors of F (iii)

$\sigma(F)$ is the sum of the absolute values of the primary divisors of F: (iv)

$$\sigma(F) = \sum_{D \mid F} |D|$$

$\varphi(F)$ is the number of polynomials of degree less than F that are prime to F. (v)

$Q(\nu)$ is the number of primary polynomials of degree ν that are not divisible by the square of an irreducible polynomial. (vi)

4. By making use of the connection between these functions and $\zeta(s)$ we deduce the following results very easily.

$$\sum_{\deg F=\nu} \mu(F) = 0 \quad \text{for } \nu \geqq 2, \tag{i}$$

$$\sum_{\deg F=1} \mu(F) = -p^{\pi}$$

$$\sum_{\deg F=\nu} \lambda(F) = (-1)^{\nu} p^{\pi[(\nu+1)/2]}, \tag{ii}$$

where $[\alpha]$ is the greatest integer $\leqq \alpha$.

$$\sum_{\deg F=\nu} \tau(F) = (\nu+1)p^{\pi\nu}. \tag{iii}$$

$$\sum_{\deg F=\nu} \sigma(F) = p^{\pi\nu} \frac{p^{\pi(\nu+1)} - 1}{p^{\pi} - 1}. \tag{iv}$$

$$\sum_{\deg F=\nu} \varphi(F) = p^{2\pi\nu} - p^{\pi(2\nu-1)}. \tag{v}$$

$$Q(\nu) = p^{\pi\nu} - p^{\pi(\nu-1)} \quad \text{for } \nu > 1, \tag{vi}$$

$$Q(1) = p^{\pi}$$

These formulas are the analogs of well-known asymptotic formulas in the rational field.

5. We now give a theorem of reciprocity of index $p^{\pi} - 1$. Define $\{A/P\}$ as that element of $GF(p^{\pi})$ such that

$$\left\{\frac{A}{P}\right\} \equiv A^{(|P|-1)/(p^{\pi}-1)}, \text{ mod } P, \quad P \text{ not dividing } A.$$

Then if P and Q are primary and prime to each other

$$\left\{\frac{Q}{P}\right\} = (-1)^{\rho\nu} \left\{\frac{P}{Q}\right\}, \tag{1}$$

ρ,ν being the degrees of Q, P, respectively.

From this Dedekind's theorem of quadratic reciprocity[1] follows as a special case.

6. The proof of (1) depends on three lemmas. Define $R(A/P)$ as the remainder in the division of A by P.

Lemma 1. (*Analog of Gauss' Lemma.*) *If H run through the primary polynomials of degree $<\nu$,*

$$\left\{\frac{A}{P}\right\} = \prod_{H} sgn\, R\left(\frac{AH}{P}\right).$$

Lemma 2. *If A is primary of degree $\alpha \geqq \nu$, and is not a multiple of P,*

$$sgn\ \Pi(A - KP) = (-1)^{\alpha - \nu}\ sgn\ R\left(\frac{A}{P}\right),$$

the product extending over all primary K of degree $\alpha - \nu$.

Lemma 3. *If H runs through the primary polynomials of degree* $< \nu$,

$$sgn \prod_{H,K} (HQ - KP) = (-1)^{\rho\nu + Min.\ (\rho^2, \nu^2)}\ sgn \prod_{H} R\left(\frac{HQ}{P}\right).$$

7. A paper containing a detailed account of the above results, as well as a number of generalizations, has been offered to the *American Journal of Mathematics*.

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[1] Cf. Dedekind, R., *J. für. Math.*, **54** (1857), pp. 1–26.

THE COLORING OF GRAPHS[1]

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1. *Introduction.*—We shall give here an outline of a study of the numbers m_{ij} appearing in a formula for the number of ways of coloring a graph. The details will be given in several papers. The definitions and results in a paper on Non-separable and Planar Graphs will be made use of.

2. *The Number of Ways of Coloring a Graph.*—Suppose we assign to each vertex of a graph a color in such a way that each pair of vertices joined by an arc are of different colors. (A graph containing a 1-circuit cannot be colored therefore.) We obtain thereby a permissible coloring of the graph. Given a graph G, let m_{ij} be the number of subgraphs of rank i and nullity j. Then the number of ways of coloring G in λ colors is

$$P(\lambda) = \sum_i \lambda^{v-i} \sum_j (-1)^{i+j} m_{ij} = \sum_i m_i \lambda^{v-i},$$

if G contains v vertices. This result, first found by Birkhoff,[2] is proved by a simple logical expansion.[3]

We note that, if G contains E arcs,

$$m_{i0} + m_{i-1,1} + \ldots + m'_{0i} = \binom{E}{i}.$$

Let G' be formed from G by dropping out the arc ab. Let $m'_{ij}(a \times b)$ be the number of subgraphs of rank i, nullity j, of G' in which a and b are in different connected pieces. Put

$$m_i'(a \times b) = \sum_j (-1)^{i+j} m_{ij}'(a \times b).$$

Then

$$m_i = m_i'(a \times b) - m_{i-1}'(a \times b).$$

Another interesting recursion formula is as follows: m_{ij} for G equals the sum of the m_{ij}'s for all the subgraphs of G formed by dropping out a single arc, divided by $i + j$, if $i + j < E$. The first recursion formula can be extended, and gives the following results:

Let us arrange the arcs of G in a definite order: $\alpha,\beta,\gamma,\ldots,\epsilon$. Given any circuit, we form the corresponding *broken circuit* by dropping out the last arc of the circuit. Thus, if α,β,γ form a circuit, α,β is a broken circuit.

THEOREM 1. *$(-1)^i m_i$ is the number of ways of picking out i arcs from G so that not all the arcs of any broken circuit are removed.*

The coefficient of λ^v in $P(\lambda)$ is 1 if G contains no 1-circuit. We can show that the coefficient of λ^{v-R} (that is, the coefficient of λ, if G is connected), is 0. This with Theorem 1, gives

THEOREM 2.

$$(-1)^i m_i > 0, \quad i = 0, 1, \ldots, R.$$

3. *The Numbers α_i and β_i.*—Let $\alpha,\beta;\ldots;\gamma,\delta,\epsilon$ be the broken circuits of G. In theorem 1, we picked out i arcs from G so that the property A holds true, that is, α and β are not picked out, $\ldots\gamma,\delta$ and ϵ are not picked out. If we expand A by the logical expansion, α_i is the number of logical terms in the result containing i arcs. If we expand A into the second normal form (see a forthcoming paper by the author), β_i is the number of logical terms in this result containing i arcs. m_i is given in terms of the numbers α_i or β_i by certain arithmetical formulas.

4. *The Numbers m_{ij} and f_{ij}.*—To find m_{ij} for a graph G, we count all its subgraphs. But this is not necessary, for it is sufficient to count only the non-separable subgraphs.

THEOREM 3. *m_{ij} is a polynomial in the numbers N_1, N_2, ..., of non-separable subgraphs of certain types in G. The same is, therefore, true of m_i.*

For example, if G contains no 1- or 2-circuits, and N_{21}, N_{31} are the number of triangles and quadrilaterals in G, then

$$m_{31} = (E - 3)N_{21} + N_{31}.$$

Now let m_{ij} be the coefficient of $x^i y^j$ in a polynomial Q, of which the constant term is $m_{00} = 1$. Thus $Q = 1 + Q'$. Expand the logarithm of this polynomial as a power series in x and y. Let the coefficient of $x^i y^j$ in this series be f_{ij}. Then f_{ij} equals m_{ij} plus a polynomial in m_{kl}, $k \leqq i$, $l \leqq j$, containing no constant or linear term, and thus m_{ij} equals a similar expression in f_{kl}. Also, from Theorem 3 we see that f_{ij} is a polynomial in the numbers N_1, N_2, ...

Let $G = G' + G''$ be the graph containing G' and G'' as separate pieces. From the definition of m_{ij}, we find that

$$m_{ij} = \sum_{k,l} m'_{kl}\, m''_{i-k,\, j-l}.$$

THEOREM 4.

$$f_{ij} = f'_{ij} + f''_{ij}.$$

THEOREM 5. *Let $T_1, \ldots, T_q$ be types of non-separable subgraphs, and let p be any positive number. Then we can find a set of numbers $N_1, \ldots, N_q$, each $N_i > 0$, such that given any set of numbers $n_1, \ldots, n_q$, $0 \leq n_i \leq p$, there exists a graph containing $N_i + n_i$ subgraphs of type T_i, $i = 1, \ldots, q$.*

From the last two theorems we can deduce the following:

THEOREM 6. *Consider the expression for m_{ij} as a polynomial in the numbers $N_1, N_2, \ldots$, of non-separable subgraphs of certain types of G_i. f_{ij} is exactly the linear terms of this polynomial.*

Thus if we can find the linear terms of the polynomial, we can, by a fixed arithmetical transformation, find the whole of m_{ij}.

As f_{ij} is linear, it has the form

$$f_{ij} = c_1 N_1 + c_2 N_2 + \ldots.$$

Is it difficult to find the coefficients $c_1, c_2, \ldots$? If $N_{k-1,1}$, is the number of k-circuits in G, the author has found the corresponding coefficients $c_{k-1,1}$ for all the numbers f_{i0}, the most interesting set. These coefficients turned out to be quite simple; it would be very worthwhile to find still more coefficients for f_{i0}.

A fundamental problem of the theory is the following: When do a set of numbers m_{ij} represent a graph? In 2 we found a relation holding between the m_{ij}.

THEOREM 7. *There exists no other polynomial relation between the m_{ij}, true for all graphs.*

However, there exist many inequalities, for instance, those induced by theorem 2. The great difficulty of the problem is now apparent.

5. *Relation to the 4-Color Map Problem.*

THEOREM 8. *Suppose G is a planar graph, and G' is its dual. Then*

$$m'_{ij} = m_{N-j,\, R-i}.$$

PROPOSITION. *Let m_{ij} be the numbers for G. If $m'_{ij} = m_{N-j,\, R-i}$ are the numbers for some graph G', then G is planar, and G' is its dual.*

This proposition, though not proved, seems very likely true. If so, we have a new statement of the 4-color map problem:

Is it true that if m_{ij} are any set of numbers such that both they and the numbers $m_{N-j,\, R-i}$ are the numbers of graphs G and G' (R and N being the rank and nullity of G), then

$$\sum_{ij} (-1)^{i+j} m_{ij} 4^{v-i} > 0?$$

More generally, this study is meant to throw some light on the nature of the numbers m_{ij}, which may in time lead to solutions of problems such as the 4-color problem.

[1] Presented to the American Mathematical Society, Oct. 25, 1930.

[2] Birkhoff, G. D., *Ann. Math.*, 2, **14**, No. 1 (42–46).

[3] See Whitney, H., *Bull. Am. Math. Soc.*, Abstract No. 36–11–396.

NON-SEPARABLE AND PLANAR GRAPHS[1]

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1. *Introduction.*—We shall give here an outline of the main results of a research on non-separable and planar graphs. The methods used are entirely of a combinatorial character; the concepts of rank and nullity play a fundamental rôle. The results will be given in detail in a later paper.

A *graph* G is composed of two sets of symbols: *vertices*, $a, b, \ldots, f$, and *arcs*, $\alpha(ab)$ (or simply ab), $\beta(ac), \ldots, \delta(ef)$. A *chain* is a set of distinct arcs and vertices, $ab, bc, \ldots, de$. A *suspended chain* is a chain containing at least two arcs, of which no vertices are on other arcs but the first and last vertices, which are each on at least two other arcs. A *circuit* is a set of distinct arcs and vertices, $ab, bc, \ldots, de, ea$. A *k-circuit* is a circuit containing k arcs. A *subgraph* H of G is a graph formed by dropping out arcs from G. Let V, E, P be the number of vertices, arcs and connected pieces in G. We define the *rank* R and the *nullity* (or cyclomatic number) N by the equations

$$R = V - P,$$
$$N = E - R = E - V + P.$$

2. *Non-Separable Graphs.*—G is called *non-separable* if it is connected, and if there are no two graphs G_1 and G_2, each containing at least one arc, which form G if a vertex of one is made to coalesce with a vertex of the other. If G is not non-separable, it is separable. G is called *cyclicly connected* if each pair of vertices is contained in a circuit in G.

THEOREM 1. *Let G be a graph containing at least two arcs but no 1-circuit. A necessary and sufficient condition that G be non-separable is that G be cyclicly connected.*[2]

Suppose a connected part of G is separable. We then separate it at a

vertex into two parts. We continue in this manner until each connected part is non-separable. We say G is decomposed into its non-separable *components*, or its components, simply.

THEOREM 2. *A graph G may be decomposed into its components in a unique manner.*

THEOREM 3. *Let $R_1, \ldots, R_m, N_1, \ldots, N_m$ be the ranks and nullities of the components $G_1, \ldots, G_m$ of G. Then*

$$R = R_1 + \ldots + R_m,$$

$$N = N_1 + \ldots + N_m.$$

THEOREM 4. *A necessary and sufficient condition that G be non-separable is that there exist no division of its arcs into two groups G_1 and G_2 so that*

$$R = R_1 + R_2.$$

We shall say two non-separable graphs form a *circuit of graphs* if they have at least two common vertices. A set of three or more non-separable graphs form a circuit of graphs if they are in cyclic order, each graph has just one vertex in common with the graph following, these vertices are distinct, and no other pair of these graphs have a vertex in common.

THEOREM 5. *Let $G_1, \ldots, G_m$ be a set of non-separable graphs, each containing at least one arc, and let G be formed by letting vertices and arcs of different of these graphs coalesce. Then the following four statements are all equivalent:*

(1) *$G_1, \ldots, G_m$ are the non-separable components of G.*

(2) *No two of the graphs $G_1, \ldots, G_m$ have an arc in common, and there is no circuit in G containing arcs of more than one of these graphs.*

(3) *No subset of these graphs form a circuit of graphs.*

(4)

$$R = R_1 + \ldots + R_m.$$

THEOREM 6. *A non-separable graph of nullity 1 is a circuit.*

THEOREM 7. *If G is a non-separable graph of nullity $N > 1$, we can remove an arc or suspended chain, leaving a non-separable graph of nullity $N-1$.*

3. *Homeomorphic Graphs.*—If we can rename the vertices and arcs of a graph G', giving distinct vertices and distinct arcs different names, so that G' becomes identical with G, we say G and G' are *homeomorphic*. If G and G' become homeomorphic when they are decomposed into their non-separable components and any isolated vertices are dropped out, we say they are *equivalent*.

THEOREM 8. *Let G and G' be two graphs containing no isolated vertices or 2 circuits. Let there exist a 1-1 correspondence between their arcs so that*

(1) *To each* 1*-circuit in one corresponds a* 1*-circuit in the other,*

(2) *To each pair of arcs in one having* i *common vertices corresponds a pair of arcs in the other having* i *common vertices,* $(i = 0, 1)$*, and*

(3) G *and* G' *contain the same number of connected pieces whose arcs are of the form* ab, ac, ad.

Then G *and* G' *are homeomorphic.*

4. *Duals, Planar Graphs.*—If H_1 is a subgraph of G, that subgraph H_2 of G containing those arcs not in H_1 is called the complement of H_1 in G. Let R, R', r, r', etc., stand for the ranks of G, G', H, H', etc., respectively, with similar definitions for nullity. Suppose there is a 1-1 correspondence between the arcs of G and G' such that if H is any subgraph of G and H' is the complement of the corresponding subgraph of G, then

$$r' = R' - n.$$

We say then G' is a *dual* of G.

THEOREM 9. *Let* G' *be a dual of* G. *Then*

$$R' = N, \quad N' = R.$$

THEOREM 10. *If* G' *is a dual of* G, G *is a dual of* G'.

THEOREM 11. *If* G' *and* G'' *are equivalent and* G' *is a dual of* G, *then* G'' *is a dual of* G.

THEOREM 12. *Let* $G_1, \ldots, G_m$ *and* $G_1', \ldots, G_m'$ *be the components of* G *and* G', *respectively, and let* G_i' *be a dual of* G_i, $i = 1, \ldots, m$. *Then* G' *is a dual of* G.

THEOREM 13. *Let* G *and* G' *be dual graphs, and let* $G_1, \ldots, G_m$ *be the components of* G. *Let* $G_1', \ldots, G_m'$ *be the corresponding subgraphs of* G'. *Then* $G_1', \ldots, G_m'$ *are the components of* G', *and* G_i' *is a dual of* G_i, $i = 1, \ldots, m$.

From this theorem follows the

THEOREM 14. *A dual of a non-separable graph is non-separable.*

Let a topological graph be a graph whose arcs are sets of points in 1-1 correspondence with the unit interval, the end vertices of the arcs corresponding with the ends of the interval. We may associate a topological graph with each graph, or abstract graph, and conversely. A topological graph is called planar if it can be mapped in a 1-1 continuous manner on a plane (or sphere). An abstract graph is planar if the corresponding topological graph is planar.

THEOREM 15. *A necessary and sufficient condition that a graph be planar is that it have a dual.*

[1] Presented to the American Mathematical Society, Oct. 25, 1930.

[2] Compare Whyburn, G. T., these PROCEEDINGS, **13**, 1927 (31–38), Theorem 1; Ayres, W. L., *Am. J. Math.*, **51**, 1929(577–594), Theorem 10.

THE PRINCIPLE OF MAUPERTUIS

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For a dynamical system whose kinetic energy is T and whose potential energy is V, Hamilton's principle tells us that the first variation of the time integral of the Lagrangian function

$$L = T - V$$

is zero along a trajectory of the system. The kinetic energy T is a quadratic function (not necessarily homogeneous) in the velocities and for a "natural" dynamical system L does not involve explicitly the time variable t. In this latter case the Principle of Maupertuis states that the Hamiltonian integral may be replaced by another which is homogeneous of the first degree in the derivatives of the coördinates with respect to the independent variable, but which involves the energy constant. It is the purpose of this note to show the real basis of this Principle of Maupertuis and to give explicitly the formulation of the integrand of Maupertuis for an arbitrary Lagrangian function which does not involve explicitly one of the coördinates.

Denoting the coördinates of a dynamical system with n degrees of freedom by $(x^1, x^2, \ldots, x^n)$ the time t may conveniently be denoted by x^0 and time derivatives will be indicated by a superposed dot. Let L be any function of the time x^0, the coördinates $x = (x^1, x^2, \ldots, x^n)$ and their time derivatives $(\dot{x}^1, \dot{x}^2, \ldots, \dot{x}^n)$. On introducing arbitrarily a new independent variable τ by means of the equation $x^0 = x^0(\tau)$ and denoting derivatives with respect to τ by a prime, the integrand in the Hamiltonian integral becomes

$$F(x^0, x, x^{0'} x') \equiv L\left(x^0, x, \frac{x'}{x^{0'}}\right).$$

F is homogeneous of degree unity in the quantities $(x^{0'}, x')$ so that the $(n + 1)$ derivatives

$$p_0 = \frac{\partial F}{\partial x^{0'}}; \quad p_r = \frac{\partial F}{\partial x^{r'}} \quad (r = 1, 2, \ldots, n)$$

(which constitute the momentum-energy covariant tensor) are homogeneous of degree zero in the $n + 1$ derivatives $(x^{0'}, x')$ and hence involve these derivatives only through their n ratios. Eliminating these n ratios we obtain *one or more* relations of the type

$$\phi(x^0, x, p_0, p) = 0. \qquad \text{(A)}$$

The homogeneous character of F enables us to write the Hamiltonian integral

$$\int L dt \equiv \int F d\tau$$

in the form

$$\int \left(x^{0'} \frac{\partial F}{\partial x^{0'}} + x^{\alpha'} \frac{\partial F}{\partial x^{\alpha'}} \right) d\tau = \int (p_0 dx^0 + p_\alpha dx^\alpha)$$

where the summation symbol α runs over the values 1 to n. The variational principle may be then stated in the $2n + 2$ dimensional (state-energy) space in which the coördinates are (x^0, x, p_0, p) but it is a constrained problem, as these coördinates must satisfy the relation (or relations) (A). The Euler-Lagrange equations for the problem so stated give (when there is only one relation) the usual canonical equations of Hamilton

$$\frac{dx^0}{d\tau} = \mu \frac{\partial \varphi}{\partial p_0}; \quad \frac{dx^r}{d\tau} = \mu \frac{\partial \varphi}{\partial p^r}; \quad \frac{dp_0}{d\tau} = -\mu \frac{\partial \varphi}{\partial x^0}; \quad \frac{dp_r}{d\tau} = -\mu \frac{\partial \varphi}{\partial x^r}, \quad \text{(B)}$$

where μ is an undetermined multiplier.

When there are s relations (A) these must be replaced by the equations

$$\frac{dx^0}{d\tau} = \underset{\alpha}{\mu} \frac{\partial^\alpha \varphi}{\partial p_0}; \quad \frac{dx^r}{d\tau} = \underset{\alpha}{\mu} \frac{\partial^\alpha \varphi}{\partial p_r}; \quad \frac{dp_0}{d\tau} = -\underset{\alpha}{\mu} \frac{\partial^\alpha \varphi}{\partial x^0}; \quad \frac{dp_r}{d\tau} = -\underset{\alpha}{\mu} \frac{\partial^\alpha \varphi}{\partial x^r},$$

where the summation label α runs from *1* to s and there are now s undetermined multipliers $(\underset{1}{\mu}, \underset{2}{\mu}, \ldots, \underset{3}{\mu})$. In the dynamical problem it is assumed that the equations

$$p_r = \frac{\partial F}{\partial \dot{x}^r} = \frac{\partial L}{\partial \dot{x}^r} \quad (r = 1, 2, \ldots, n)$$

may be solved for the n quantities $\dot{x}^r$ in terms of the n quantities p_r and on substituting these in the equation

$$p_0 = \frac{\partial F}{\partial \dot{x}^0} = L(x^0, x, \dot{x}) - \dot{x}^\alpha p_\alpha$$

we obtain the *single* relation

$$\varphi(x^0, x, p_0, p) \equiv p_0 + H(x^0, x, p) = 0$$

where H is the usual Hamiltonian function. On using this particular form for φ it is clear that the canonical equations (B) take the familiar form

$$\frac{dx^0}{1} = dx^r \frac{\partial H}{\partial p_r}\Big/ = dp_0 \Big/ \frac{-\partial H}{\partial x^0} = dp_r \Big/ \frac{-\partial H}{\partial x^r} = \mu .$$

Let us now consider, conversely, a scalar function $\varphi(x^0, x, p_0, p)$ of the coördinates (x^0, x) and a covariant vector (p_0, p) which has the value zero:

$$\varphi(x^0, x, p_0, p) = 0. \qquad \text{(C)}$$

The $n + 1$ partial derivatives $\left(\frac{\partial \varphi}{\partial p_0}, \frac{\partial \varphi}{\partial p_r}\right)$ constitute a contravariant vector and we write

$$\frac{\partial \varphi}{\partial p_0} = \frac{\lambda^0}{k}; \qquad \frac{\partial \varphi}{\partial p_r} = \frac{\lambda^r}{k} \qquad \text{(D)}$$

where k is an undetermined factor of proportionality. Assuming that the determinant $\left|\frac{\partial^2 \varphi}{\partial p_i \partial p_j}\right|$ $(i, j = 0, 1, \ldots, n)$ does not vanish identically these equations may be solved for (p_0, p) in terms of the λ/k and on substituting these in (C) the expression for k in terms of $(x^0, x, \lambda^0, \lambda)$ follows. When this is done (p_0, p) appear as homogeneous functions of degree zero in the (λ^0, λ). Denoting the scalar product $\lambda^\alpha p_\alpha$ (where the summation label runs over the values 0, 1, 2, ..., n) by S:

$$S = \lambda^\alpha p_\alpha$$

we have $dS = \lambda^\alpha dp_\alpha + p_\alpha d\lambda^\alpha$. On differentiating (C) and using (D) we have

$$0 = \frac{\partial \varphi}{\partial x^\alpha} dx^\alpha + \frac{\partial \varphi}{\partial p_\alpha} dp_\alpha = \frac{\partial \varphi}{\partial x^\alpha} dx^\alpha + \frac{\lambda^\alpha}{k} dp_\alpha$$

so that $dS = p_\alpha d\lambda^\alpha - k \frac{\partial \varphi}{\partial x^\alpha} dx^\alpha$. As the variables x and λ are independent this gives

$$\frac{\partial S}{\partial \lambda^0} = p_0; \quad \frac{\partial S}{\partial x_r} = p_r; \quad \frac{\partial S}{\partial x^0} = -k \frac{\partial \varphi}{\partial x^0}; \quad \frac{\partial S}{\partial x^r} = -k \frac{\partial \varphi}{\partial x^r} . \qquad \text{(E)}$$

On comparing these equations with the equations (B) we see that if we set

$$\lambda^0 = \frac{k}{\mu} \frac{dx^0}{d\tau}; \lambda^r = \frac{k}{\mu} \frac{dx^r}{d\tau}$$

we have

$$\frac{d}{d\tau}\left(\frac{\partial S}{\partial x^{0'}}\right) = \frac{\partial S}{\partial x^0}; \quad \frac{d}{d\tau}\left(\frac{\partial S}{\partial x^{r'}}\right) = \frac{\partial S}{\partial x^r},$$

so that S is the homogeneous Lagrangian function corresponding to the Hamiltonian relation $\varphi(x^0, x, p_0, p) = 0$. In particular, if φ has the form $p_0 + H(x^0, x, p)$ we have $\frac{\partial \varphi}{\partial p_0} = 1$ so that $k = \lambda^0$ and the equations

$$\frac{\partial H}{\partial p_r} = \frac{\lambda^r}{\lambda^0}$$

serve to determine $(p_1, \ldots, p_r)$ in terms of the ratios $\frac{\lambda^r}{\lambda_0}$. The remaining component p_0 follows from $p_0 + H(x^0, x, p) = 0$.

When one of the coördinates (x^0, x) is missing from $\varphi(x^0, x, p_0, p)$ (i.e., is ignorable) the equations (B) show that the corresponding momentum is constant and the two equations of the set (B) which correspond to this coördinate and momentum may be omitted when solving for the remaining n x's and n p's. The Principle of Maupertuis is merely the application of this observation to the case where the ignorable coördinate is the time x^0 and the Lagrangian function has the form $L = T - V$ where T (in the usual formulation) $= {}^1/_2\, g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ and V is a function of $(x^1, \ldots, x^n)$. Then

$$\varphi \equiv p_0 + H(x, p)$$

where $$H(x, p) = {}^1/_2\, g^{\alpha\beta}p_\alpha p_\beta + V.$$

On setting $\frac{\partial \varphi}{\partial p_r} = \frac{\lambda^r}{k}$ we obtain $g^{r\alpha}p_\alpha = \frac{\lambda^r}{k}$ so that $p_r = g_{r\alpha}\frac{\lambda^\alpha}{k}$. Direct substitution in $\varphi = 0$ gives

$$p_0 + {}^1/_2\, g_{\alpha\beta}\frac{\lambda^\alpha\lambda^\beta}{k^2} + V = 0 \text{ so that } k = \sqrt{\frac{g_{\alpha\beta}\lambda^\alpha\lambda^\beta}{-2(p_0 + V)}}$$

The homogeneous Lagrangian integrand is accordingly

$$S = \lambda^\alpha p_\alpha = \frac{g_{\alpha\beta}\lambda^\alpha\lambda^\beta}{k} = \sqrt{g_{\alpha\beta}\lambda^\alpha\lambda^\beta}\ \sqrt{-2(p_0 + V)}.$$

Writing $\lambda^r = \frac{dx^r}{d\tau}$ and $p_0 = -h$ this gives the usual statement of the Principle of Maupertuis. It is sufficiently evident from what we have written that the force of the principle is this: When a problem in the calculus of variations (one independent variable, n dependent variables), has a non-homogeneous integrand the problem can always be transformed

so as to have a homogeneous integrand but at the expense of increasing the number of dependent variables by one (from n to $n + 1$). However when there is an ignorable coördinate the problem can be transformed into one with a homogeneous integrand with only n dependent variables, this integrand involving the constant momentum which corresponds to the ignorable coördinate. Since the parameter can be chosen arbitrarily when the integrand is homogeneous this enables us to take one of the coördinates themselves as the independent variable thus reducing the problem to one with $n - 1$ (instead of n) dependent variables.

NOTES ON SCALAR EXTENSIONS OF TENSORS AND PROPERTIES OF LOCAL COÖRDINATES

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1. The differential geometry of the n-dimensional group manifold of an n-parameter continuous group of transformations was, I believe, first studied by Cartan[1] and Schouten[1] in 1926. This group manifold possesses two *teleparallelisms*[2] (distant parallelisms). There results a unique symmetric affine connection $\Gamma^{\alpha}_{\beta\gamma}$

$$\Gamma^{\alpha}_{\beta\gamma} = {}^1/_2\, {}_i\xi^{\alpha}\left(\frac{\partial {}^i\xi_{\beta}}{\partial x^{\gamma}} + \frac{\partial {}^i\xi_{\gamma}}{\partial x^{\beta}}\right), \quad \left|\, {}_i\xi^{\alpha}\,\right| \neq 0. \tag{1}$$

Einstein's[2] convention for Greek and Latin indices as well as his summation convention will be employed throughout our paper. A few exceptions will occur (such as those of formula (5) and (10) below) but this need not cause any confusion. Now the ennuple of contravariant vectors ${}_i\xi^{\alpha}$ (the associated ennuple of covariant vectors is denoted by ${}^i\xi_{\alpha}$) is not arbitrary but satisfies the differential equations

$${}_i\xi^{\sigma}\frac{\partial\, {}_j\xi^{\alpha}}{\partial x^{\sigma}} - {}_j\xi^{\sigma}\frac{\partial\, {}_i\xi^{\alpha}}{\partial x^{\sigma}} = c^{k}_{ij}\, {}_k\xi^{\alpha}, \tag{2}$$

where c^{k}_{ij} are the structural constants of the group.

In the September, 1928, issue of these Proceedings,[3] I published a paper entitled "The Group Manifold of Finite Continuous Point and Functional Transformation Groups." In this paper I developed the subject in such a manner as to bring into the foreground the various fundamental invariants in an *arbitrary coördinate system*. This was achieved with the

aid of a *local coördinate system* z^i that was defined implicitly in terms of an arbitrarily chosen coördinate system x^α by

$$x^\alpha = q^\alpha + {}_i\xi^\alpha(q)z^i - \frac{1}{2!}\Gamma^\alpha_{\lambda\mu}(q)\,{}_i\xi^\lambda(q)\,{}_j\xi^\mu(q)\,z^i z^j, \tag{3}$$

where q^α are the coördinates in the x^α system of the origin of the z^i coordinates such a coördinate system z^i has the property that

$$\left.\begin{aligned} {}_i\bar{\xi}^\alpha(z) = {}_i\delta^\alpha \text{ and } \bar{\Gamma}^\alpha_{\lambda\mu}(z) = 0 \\ ({}_i\delta^\alpha = 0 \text{ or } 1 \text{ according as } \alpha \neq i \text{ or } \alpha = i) \end{aligned}\right\} \tag{4}$$

at the origin $z^i = 0$. The bar over ξ and Γ denotes that the functions are referred to a z^i coördinate system and then evaluated in that system. In the paper A, I used the terminology canonical (instead of local) coordinate system for any system z^i that satisfies the above conditions (4). *The relationship* (3) *between the local coördinates* z^i *and the arbitrary coordinates* x^α *was used repeatedly to demonstrate theorems in the group manifold.* For example the first *tensor* extension ${}_j\xi^\alpha_{,\beta}$ (covariant derivative of ${}_j\xi^i_{,k}$) was derived by first obtaining the *first scalar extension* ${}_j\xi^i_{,k}$ of the group vectors ${}_j\xi^\alpha$

$${}_j\xi^i_{,k} = \left(\frac{\partial\,{}_j\bar{\xi}^i}{\partial z^k}\right)_{z=0}. \tag{5}$$

It turns out that

$${}_j\xi^i_{,k} = {}^1/_2 c^i_{kj}\,;\; {}_j\xi^\alpha_{,\beta} = {}^1/_2 c^i_{kj}\,{}_i\xi^\alpha\,{}^k\xi_\beta.$$

For further details of this particular example see § 5 of paper A.

2. The special results on local coördinates and scalar extensions in paper A have more recently led me to the consideration of a general and systematic theory of local coördinates and scalar extensions. In fact, in the March, 1930, issue of the *Bulletin of the American Mathematical Society*[4] I published a brief note entitled "An Operation That Generates Scalar Differential Invariants from Tensor." A complete account of this subject is still under press.[5] A special situation has been singled out and discussed in another paper.[6] I shall be content in the present paper with the statement of two fundamental theorems that were proved in paper B. In §3 I consider two theorems (3 and 4) that are not contained in paper B.

Let ${}_i\xi^\alpha$ be an ennuple of linearly independent vectors and let our n dimensional space be further endowed with a symmetric affine connection $\Gamma^\alpha_{\beta\gamma}$. These two sets of functions need not be of the special type used in the theory of the group manifold. The ${}_i\xi^\alpha$ may be, for example, the orthogonal ennuple used by Einstein[2] in his "Unified Field Theory" or what not.

Definition.—A coördinate system z^i that is defined implicitly by

$$\left.\begin{aligned} x^\alpha = {} & q^\alpha + {}_i\xi^\alpha(q)\, z^i \\ & - \sum_{s=1}^{r} \frac{1}{(s+1)!}\, \Gamma^\alpha_{\lambda\mu_1\ldots\mu_s}\, {}^{(q)}_{i}\xi^\lambda_{(q)}\, {}^{\mu_1}_{j_1}\xi^{\mu_1}_{(q)} \cdots {}^{\mu_s}_{j_s}\xi^{\mu_s}_{(q)}\, z^i z^{j_1} \ldots z^{j_s} \end{aligned}\right\} \qquad (6)$$

will be called a coördinate system S_r (a local coördinate system of order r).

For definitions of the "generalized affine connections" $\Gamma^\alpha_{\lambda\mu_1\ldots\mu_s}$ see a recent paper.[7] With the aid of the relations (6) and

$$({}_i\bar{\xi}^\alpha(z))_{z=0} = {}_i\delta^\alpha\,, \quad (\bar{\Gamma}^\alpha_{\lambda\mu_1\ldots\mu_s}(z))_{z=0} = 0 \qquad (s = 1, 2, \ldots, r) \qquad (7)$$

one can prove the following theorems.

THEOREM 1. *Any two coördinate systems z^i and $\bar{z}^i$ of type S_r with the same origin are related by means of a transformation*

$$\bar{z}^i = z^i + h^i_{r+2}(z) \qquad (8)$$

where $h^i_{r+2}(z)$ is a regular power series in z^i of the form

$$h^i_{r+2}(z) = \sum_{s=r+2}^{\infty} a^i_{j_1\ldots j_s}\, z^{j_1} \ldots z^{j_s}. \qquad (9)$$

THEOREM 2. *If z^i is a coördinate system of type S_r and if $T^{\alpha\ldots\beta}_{\gamma\ldots\delta}$ are the components of a tensor, then the functions $t^{i\ldots j}_{k\ldots l,\, m_1\ldots m_r}(x^1, \ldots, x^n)$* defined by

$$t^{\alpha\ldots\beta}_{\gamma\ldots\delta,\, \epsilon_1\ldots\epsilon_r}(q) = \left(\frac{\partial^r \bar{T}^{\alpha\ldots\beta}_{\gamma\ldots\delta}(z)}{\partial z^{\epsilon_1}\ldots\partial z^{\epsilon_r}}\right)_{z=0} \qquad (10)$$

are absolute scalars (the rth scalar extension of T). The explicit expression for the rth scalar extension of is given by

$$t^{i\ldots j}_{k\ldots l,\, m_1\ldots m_r} = T^{\alpha\ldots\beta}_{\gamma\ldots\delta,\, \epsilon_1\ldots\epsilon_r}\, {}_k\xi^\gamma \ldots {}_l\xi^\delta\, {}_{m_1}\xi^{\epsilon_1} \ldots {}_{m_r}\xi^{\epsilon_r}\, {}^i\xi_\alpha \ldots {}^j\xi_\beta, \qquad (11)$$

where $T^{\alpha\ldots\beta}_{\gamma\ldots\delta,\, \epsilon_1\ldots\epsilon_r}$ is the rth tensor extension[8] of $T^{\alpha\ldots\beta}_{\gamma\ldots\delta}$.

In these Theorems as well as in Theorems 3 and 4 below we understand that merely the minimum hypotheses of continuity and differentiability of the functions concerned (as functions of the real variables $x^1, x^2, \ldots, x^n$) are satisfied. For example, in Theorem 2 we assume: (*a*) the existence and continuity of the derivatives of T up to and including the rth; (*b*) the existence and continuity of the derivatives of Γ up to and including the $(r-1)$st; (*c*) the continuity of ξ (of course in case Γ is of type (1), we assume the existence and continuity of the derivatives of ξ up to and including the rth).

In the November, 1930, issue of these PROCEEDINGS,[9] T. Y. Thomas published an interesting and illuminating paper entitled "On the Unified Field Theory. I." Throughout his paper, *scalar extensions* (called *absolute*

derivatives by Thomas) play a fundamental role. Thomas bases his operations on local coördinates of infinite order and thus restricts himself to the indefinite differentiability of the affine connection.

3. The purpose of this paragraph is to state and outline the proofs of two theorems.

THEOREM 3. *A linear transformation with constant coefficients of the vector base* ${}_i\xi^\alpha$

$$ {}_i\xi^\alpha_* = \alpha^j_i \, {}_j\xi^\alpha \, , \; |\alpha^i_j| \neq 0 \tag{12} $$

induces the linear transformation

$$ z^i = \alpha^i_j \, z^j_* \tag{13} $$

in the local coördinates z^i *of type* S_r *and with the same arbitrarily chosen origin whenever the affine connection belongs to one of the following three types:*

(1) $\Gamma^\alpha_{\beta\gamma}$ *independent of* ${}_i\xi^\alpha$

(2) $\Gamma^\alpha_{\beta\gamma} = {}^1/_2 \, {}_i\xi^\alpha \left(\dfrac{\partial \, {}_i\xi_\beta}{\partial x^\gamma} + \dfrac{\partial \, {}_i\xi_\gamma}{\partial x^\beta} \right)$

(3) *Christoffel symbols of the type considered by Levi-Civita*[10] *in his treatment of Einstein's*[2] *Riemann spaces with teleparallelism. In this case we restrict ourselves to orthogonal transformations* (12).

The theorem is obvious for affine connections of type (1). By calculation we obtain

$$ {}_*\Gamma^\alpha_{\beta\gamma_1 \ldots \gamma_s} = \Gamma^\alpha_{\beta\gamma_1 \ldots \gamma_s} \; (s = 1, 2, \ldots, r) $$

and as before the theorem for type (2) follows readily. A similar proof can be given for affine connections of type (3) when we restrict ourselves to orthogonal transformations (12).

THEOREM 4. *If a local coördinate system* S_r *is based on the affine connection* (2) *of Theorem* 3, *then the scalar extension* ${}^i\xi_{j,k\ldots l}$ *satisfies the identity*

$$ S({}^i\xi_{j,\,k_1 \ldots k_r}) = 0 \tag{14} $$

where S *denotes the sum of all terms obtainable from the one in the parenthesis by a cyclic permutation of the subscripts* $j, k_1, \ldots, k_r$. *Moreover, the identity* (14) *persists not only under coördinate transformations but also under arbitrary transformations* (12) *of the vector base* ${}_i\xi^\alpha$.

Now (14) can be proved by identifying

$$ S({}^i\bar{\xi}_{j,\,k_1 \ldots k_r})_{z=0} = 0 $$

with

$$ (\bar{\Gamma}^i_{jk_1 \ldots k_r})_{z=0} = 0 $$

except for a numerical factor. This can be verified with the aid of (7),

the particular form of $\Gamma^{\alpha}_{\beta\gamma}$, the recurrence relations connecting the "generalized affine connections,"[7] and the fact that the relations

$$(\overline{\Gamma}^{i}_{jk_1\ldots k_s})_{z=0} = 0 \quad (s = 1, 2, \ldots, r)$$

are equivalent to

$$S\left(\frac{\partial\,\overline{\Gamma}^{i}_{jk_1\ldots k_{s-1}}}{\partial z^{s}}\right)_{z=0} = 0 \quad (s = 1, 2, \ldots, r).$$

Finally the last part of the theorem is easily proved by a slight variation of an argument given by Thomas.[9]

Much that we have said in these fragmentary notes finds application to the *group manifold* of an n-parameter continuous Lie group.

[1] E. Cartan and J. A. Schouten, *Proc. Amsterdam Acad.*, **29** (1926). See also L. P. Eisenhart, these PROCEEDINGS, **11** (1925), pp. 246–250.

[2] A. Einstein, *Math. Ann.*, **102** (1930), pp. 685–697, as well as Einstein's papers in the *Sitzungsberichte Preuss. Ak.* (1928–1930).

[3] A. D. Michal, these PROCEEDINGS, **14** (1928), pp. 746–754. We shall refer to this as paper A. The three lines ending with formula (22) in *A* should be replaced by

$$(P^{i}_{jkl})_0 = \left(\frac{\partial P^{i}_{jk}}{\partial a^{l}} - \frac{\partial P^{i}_{jl}}{\partial a^{k}}\right)_0 + \frac{1}{4}\,c^{i}_{\alpha l}\,c^{\alpha}_{jk} - \frac{1}{4}\,c^{i}_{\alpha k}\,c^{\alpha}_{jl}.$$

With the aid of (6) and (21) we get

$$\left(\frac{\partial P^{i}_{jk}}{\partial a^{l}} - \frac{\partial P^{i}_{jl}}{\partial a^{k}}\right)_0 = -\frac{1}{4}\,c^{i}_{\alpha l}\,c^{\alpha}_{jk} + \frac{1}{4}\,c^{i}_{\alpha k}\,c^{\alpha}_{jl}. \qquad (22)$$

[4] A. D. Michal, *Bull. Amer. Math. Soc.*, **36** (1930), pp. 222–223. We shall refer to this in conjunction with the paper given in reference 5 below as paper B.

[5] A. D. Michal, "An Operation That Generates Absolute Scalar Differential Invariants from Tensors," *Tôhoku Math. J.* (1931).

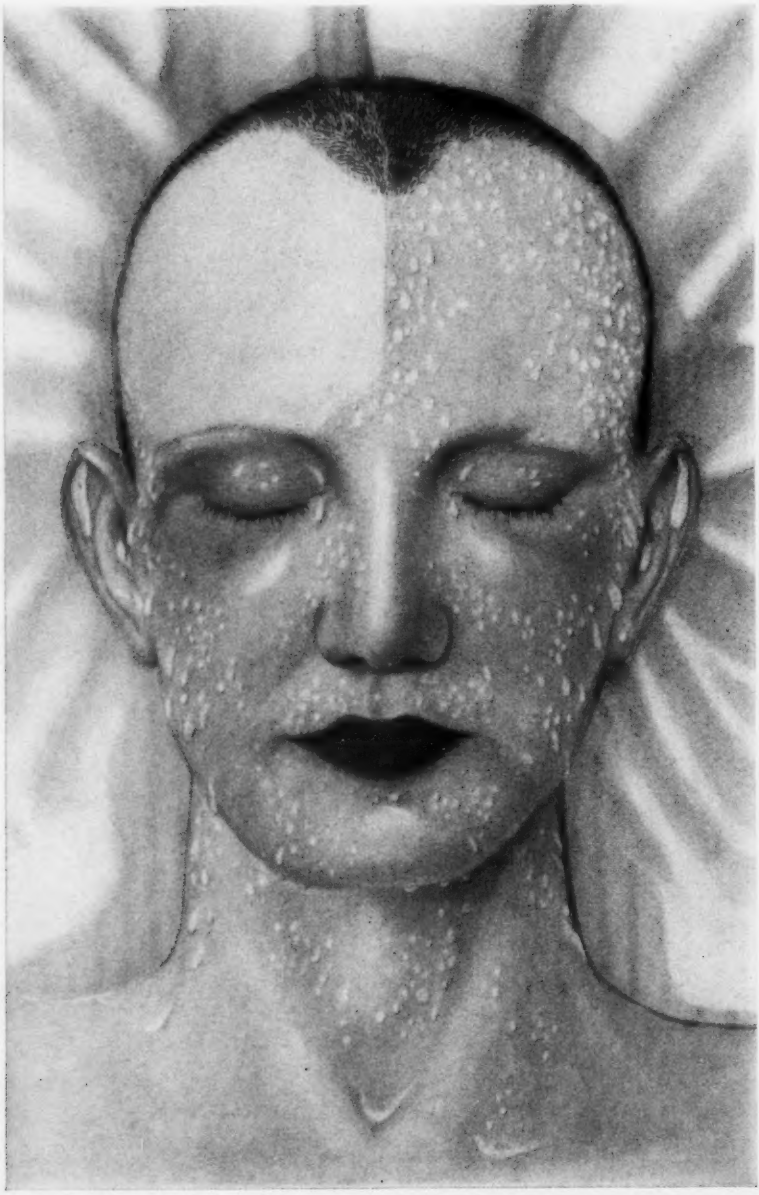
[6] A. D. Michal, "Scalar Extensions of an Orthogonal Ennuple of Vectors," *Amer. Math. Monthly*, **37** (1930), pp. 529–533.

[7] A. D. Michal, "Geodesic Coördinates of Order, r," *Bull. Amer. Math. Soc.*, **36** (1930), pp. 541–546. References to O. Veblen's and T. Y. Thomas' work on the "generalized affine connections" will be found in this paper.

[8] The *tensor* extensions of a tensor based on a symmetric affine connection $\Gamma^{\alpha}_{\beta\gamma}$ have been considered by me on the basis of the properties of geodesic coördinates of order r, a class of geodesic coördinates distinct from those of type S_r. See my paper in reference 7 above.

[9] T. Y. Thomas, these PROCEEDINGS, **16** (1930), pp. 761–776.

[10] T. Levi-Civita, *Sitzungsberichte Preuss. Ak.* (1929), pp. 137–153. See also my paper in the *Amer. Math. Monthly*, loc. cit.



Showing the vaso-dilator and sudorific effects (sparing the bone flap) of 2.5 milligrams of pilocarpine injected into the cerebral ventricles; an intraventricular injection of a cubic centimeter of pituitrin gives the same response.